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(See outside back cover.)

A MATHEMATICAL THEORY OF SWITCHING CIRCUITS

William Nemitz and Roy Reeves

The purpose of this expository paper is to present an elementary theory of switching which depends only on the properties of the elements of the circuit itself and not on an analogy between properties of circuits and properties of sets. The discussion was made purposely self contained and simple, and the results are not new but they are thought to be presented in a novel way.

The mathematical study of switching circuits is primarily concerned with an attempt to describe the characteristics of electrical circuits made up of relays or switches. We shall speak primarily of relays, although the same theory will apply to any case where one is concerned with whether or not a given electrical system conducts current, and not with quantities of current. In other words, we shall be concerned with those questions concerning conduction in electrical networks which can be answered by "yes" or "no".

It is assumed that the reader is familiar with the basic ideas of switching circuit construction. Circuits consist of conductors connected to relays or to other conductors. A contact is a point at which one conductor is connected, either to another conductor, or to a relay. In this analysis we shall be concerned with whether or not a given contact is connected to a voltage source by a continuous path of conductors in the circuit.

We shall say that two contacts, A and B, are connected in series if the following statement is true: A is connected to a voltage source if, and only if, B is connected to a voltage source.

We shall say that two contacts, A and B, are connected in parallel to a third contact, C, if both of the following statements are true:

If C is connected to a voltage source, then both A and B are connected to voltage sources. If A or B is connected to a voltage source, then C is connected to a voltage source. Note that if A and B are connected in parallel to C, then A is connected in series to C and B is connected in series to C.

A relay consists of four contacts: a pickup P, a normal output N, an input J, and a transfer output T, all related by the following rules independent of any other conductors of the circuit:

If P is connected to a voltage source, then J and T are connected in series. If P is not connected to a voltage source, then J and N are

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connected in series.

We shall say that a contact connected to a voltage source is in the connected state, and that a contact not connected to a voltage source is in the unconnected state. A contact will be called an independent contact of the circuit if its state is completely arbitrary. If the state of a contact depends on the state of any other contact of the circuit, it is a dependent contact. The state of every contact in the circuit is determined when the states of the independent contacts are determined. The state of a contact in a circuit of n independent contacts is therefore a function of the states of the n independent contacts. (We assume that we are dealing with circuits such that for given states of the n independent contacts, all contacts have a fixed single state, in other words the contacts have "stable states." An example of a circuit not satisfying this requirement is the doorbell circuit).

We may describe the state of a contact numerically by saying that if the contact is in the connected state, its state is 1, and if it is in the unconnected state, its state is 0. We may order the n independent contacts of a circuit and once this is done the states of the n independent contacts may be written as a binary n-tuple. Thus the state of any contact is described by a binary function of a binary n-tuple. We now study binary functions of binary n-tuples.

Consider the class F_n of real valued functions f of n variables a_1, $a_2, \cdots, a_n$; $f = f(a_1, a_2, \cdots, a_n)$ such that:

1) Each a_i takes the value 0 or 1.

2) For each binary n-tuple $(a_1, \cdots, a_n)$, $f(a_1, \cdots, a_n)$ is either 0 or 1.

Thus each f in F_n can be defined by a table of values, and each table of values defines a function.

Note : Any function f of m variables (m less than or equal to n) may be considered as a function of n variables. Therefore we may say F_m is contained in F_n, for m less than or equal to n. In particular the n variables $a_1, \cdots, a_n$ may be considered as functions in F_n.

Let f and g be functions in F_n. We define $f+g$, $f \cdot g$, and $f \times g$, by the following table :

f	g	$f+g$	$f \cdot g$	$f \times g$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	0
1	1	1	1	0

We define I to be that function in F_n such that for any binary n-tuple

$(a_1, \cdots, a_n)$, $I(a_1, \cdots, a_n) = 1$, and define ϕ to be that function in F_n such that for any binary n-tuple $(a_1, \cdots, a_n)$, $\phi(a_1, \cdots, a_n) = 0$. Obviously, $f + g$, $f \cdot g$, and $f \times g$ are in F_n if f and g are in F_n.

We may easily verify the following rules by the construction of tables of values:

$$(f \times g) \cdot (h \times k) = (f + h) \times (g \cdot k),$$
$$(f + g) + h = f + (g + h),$$
$$f + g = g + h,$$
$$(f \cdot g) \cdot h = f \cdot (g \cdot h),$$
$$f \cdot g = g \cdot f,$$
$$(f \cdot g) \cdot (f \times h) = \phi,$$
$$(f \cdot g) + (f \times g) = g,$$
$$(f \times g) \cdot (f \times h) = f \times (g \cdot h),$$
$$(f \cdot g) \cdot (f \cdot h) = f \cdot (g \cdot h),$$
$$f \times \phi = f \cdot \phi = \phi.$$

Here f, g, k, and h, are not necessarily distinct functions in F_n. Also, the above relationships hold true regardless whether f, g, k, and h represent functions or simply one of the numbers 0, and 1.

Note: There are exactly 2^{2^n} functions in F_n.

We now define the Triangle class of order n, of functions, designated by T_n. T_n is a subset of F_n, and is defined inductively as follows:

1) $T_0 = I$.

2) Let T_j, $j \geq 0$, be the set of functions J_{jk}, where $J_{jk}(a_1, \cdots, a_n)$ is in F_n. Then $T_{j+1} = \{a_{j+1} \cdot J_{jk}\} \cup \{a_{j+1} \times J_{jk}\}$, where J_{jk} ranges over all functions in T_j.

There are 2^j functions in T_j.

Theorem 1. For a given binary n-tuple $(a_1, \cdots, a_n)$,

$$\sum_{k=1}^{2^n} J_{nk}(a_1, \cdots, a_n) = I, \quad \left(\sum_{k=1}^{2^n} J_{nk} = J_{n1} + J_{n2} +, \cdots, + J_{n2^n} \right).$$

Proof by induction. $n = 0$, obvious. $n = 1$, $(a_1 \cdot I) + (a_1 \times I) = I$. Assume $n > 1$, and for all $m < n$, $\displaystyle\sum_{k=1}^{2^m} J_{mk}(a_1, \cdots, a_m) = I$.

Then

$$\sum_{k=1}^{2^n} J_{nk} = \left(\sum_{k=1}^{2^{n-1}} (a_n \cdot J_{n-1k}(a_1, \cdots, a_{n-1})) \right) + \left(\sum_{k=1}^{2^{n-1}} (a_n \times J_{n-1k}(a_1, \cdots, a_{n-1})) \right)$$

$$= \sum_{k=1}^{2^{n-1}} \left((a_n \cdot J_{n-1k}(a_1, \cdots, a_{n-1})) + (a_n \times J_{n-1k}(a_1, \cdots, a_{n-1})) \right)$$

$$= \sum_{k=1}^{2^{n-1}} J_{n-1k}(a_1, \cdots, a_{n-1}) = I.$$

Theorem 2. For a given binary n-tuple, $(a_1, \cdots, a_n)$, and for $n \geq 1$, $J_{nk} \cdot J_{nm} = \phi$ for $k \neq m$. (For readability, functions will be written without n-tuples in this theorem).

Proof by induction. $n = 1$, $(a_1 \cdot I) \cdot (a_1 \times I) = \phi$. Assume $n > 1$, and for all $n' < n$, $J_{n'k} \cdot J_{n'm} = \phi$ for $k \neq m$.

  Case 1. $J_{nk} = a_n \cdot J_{n-1p}$, $J_{nm} = a_n \times J_{n-1q}$. Then
$$J_{nk} \cdot J_{nm} = (a_n \cdot J_{n-1p}) \cdot (a_n \times J_{n-1q}) = \phi.$$

  Case 2. $J_{nk} = a_n \times J_{n-1p}$, $J_{nm} = a_n \times J_{n-1q}$, $p \neq q$. Then
$$J_{nk} \cdot J_{nm} = (a_n \times J_{n-1p}) \cdot (a_n \times J_{n-1q}) = a_n \times (J_{n-1p} \cdot J_{n-1q}) = a_n \times \phi = \phi.$$

  Case 3. $J_{nk} = a_n \cdot J_{n-1p}$, $J_{nm} = a_n \cdot J_{n-1q}$, $p \neq q$. Then
$$J_{nk} \cdot J_{nm} = (a_n \cdot J_{n-1q}) \cdot (a_n \cdot J_{n-1q}) = a_n \cdot (J_{n-1p} \cdot J_{n-1q}) = a_n \cdot \phi = \phi.$$

Theorem 3. For any J_{nk} in T_n, $n \geq 1$, and any two different binary n-tuples, $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$, $J_{nk}(a_1, \cdots, a_n) \cdot J_{nk}(b_1, \cdots, b_n) = \phi$.

Proof by induction. $n = 1$, $J_{1k}(a_1) \cdot J_{1k}(b_1) = \phi$, may be shown by the construction of tables of values. Assume $n > 1$, and for all $m < n$, and for all $k \leq 2^m$, $J_{mk}(a_1, \cdots, a_m) \cdot J_{mk}(b_1, \cdots, b_m) = \phi$.

  Case 1. $J_{nk} = a_n \cdot J_{n-1p}$. Then
$$J_{nk}(a_1, \cdots, a_n) \cdot J_{nk}(b_1, \cdots, b_n) = (a_n \cdot J_{n-1p}(a_1, \cdots, a_{n-1})) \cdot (b_n \cdot J_{n-1p}(b_1, \cdots, b_{n-1}))$$

$$= (a_n \cdot b_n) \cdot (J_{n-1p}(a_1, \cdots, a_{n-1}) \cdot J_{n-1p}(b_1, \cdots, b_{n-1})) = \phi.$$

  Case 2. $J_{nk} = a_n \times J_{n-1p}$. Then
$$J_{nk}(a_1, \cdots, a_n) \cdot J_{nk}(b_1, \cdots, b_n) = (a_n \times J_{n-1p}(a_1, \cdots, a_{n-1})) \cdot (b_n \times J_{n-1p}(b_1, \cdots, b_{n-1}))$$

$$= (a_n + b_n) \times (J_{n-1p}(a_1, \cdots, a_{n-1}) \cdot J_{n-1p}(b_1, \cdots, b_{n-1})) = \phi.$$

Now $f \cdot g$ is 1 if, and only if, f is 1 and g is 1. Hence by the three above theorems, given any binary n-tuple $(a_1, \cdots, a_n)$, there is one and only one J_{nk} in T_n such that $J_{nk}(a_1, \cdots, a_n) = 1$, and if $(b_1, \cdots, b_n)$ is any other binary n-tuple, then for this particular J_{nk}, $J_{nk}(b_1, \cdots, b_n) = 0$. Call this J_{nk} the J_{nk} determined by $(a_1, \cdots, a_n)$. Therefore, given any function f in F_n, there is a subset $T_{n'}$ of T_n such that $f = \Sigma J_{nk}$, the summation being taken over all of $T_{n'}$. This representation can be found as follows: For a given binary n-tuple $(a_1, \cdots, a_n)$, if $f(a_1, \cdots, a_n) = 1$, include the unique J_{nk}, determined by $(a_1, \cdots, a_n)$ as described above in $T_{n'}$. Thus $T_{n'}$ will consist of those J_{nk} for whose corresponding n-tuples $(a_1, \cdots, a_n)$, $f(a_1, \cdots, a_n) = 1$. (The corresponding n-tuple of J_{nk} is that unique n-tuple $(a_1, \cdots, a_n)$ for which $J_{nk}(a_1, \cdots, a_n) = 1$.) It should be noted that each $T_{n'}$ determines a function and each function has a corresponding $T_{n'}$. Also unequal functions will have different $T_{n'}$.

We shall call the function describing the state of a contact its state function. An examination of the table (page 2) shows the following: If A and B are two contacts connected in parallel to a third contact C, and a^*, b, and c, are the state functions of A, B, and C, respectively, and if a and b are known, then $c = a + b$. If A and B are two contacts connected in series, then $a = b$.** If P, N, J, and T, are the contacts of a relay (page 1) with state functions p, n, j, and t, and if p and j are known, then $t = p \cdot j$, and $n = p \times j$.

We assume the reader is familiar with the "christmas tree" circuit, which allows the selection of one of 2^n alternatives from n "bits" of information. A diagram of such a circuit for $n = 3$ is given. (See Figure 1). It will be noted that each output contact of a christmas tree circuit has as its state function one of the functions of T_n, and each function of T_n corresponds to a unique output contact of the christmas tree circuit. Thus, from the remarks on this page, we know that, given any binary function of

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*Lower case letters will be used for the state functions of contacts designated by the corresponding capital letters.

**The equations $c = a + b$ and $a = b$ are assumed to hold for a given set of values of the independent variables, but are not necessarily identities. Thus, for example, the question as to whether or not two contacts are in series depends not only upon the physical arrangement of the elements but upon the state of the circuit as well.

binary n-tuples, one can construct a circuit having a contact having the given function for its state function, by connecting the proper contacts of the christmas tree circuit in parallel, according to the rule given on page 5. Thus one can construct a circuit of n independent contacts having a dependent contact with any desired state function.

FIGURE 1.

A_1	A_2	A_3	J_{3k}	(a_1,a_2,a_3)
P	P	P		
		N	$a_3\times(a_2\times(a_1\times I))$	$(0,0,0)$
	N	J		
		T	$a_3\cdot(a_2\times(a_1\times I))$	$(0,0,1)$
N	J			
		N	$a_3\times(a_2\cdot(a_1\times I))$	$(0,1,0)$
	T	J		
		T	$a_3\cdot(a_2\cdot(a_1\times I))$	$(0,1,1)$
I	J			
		N	$a_3\times(a_2\times(a_1\cdot I))$	$(1,0,0)$
	N	J		
		T	$a_3\cdot(a_2\times(a_1\cdot I))$	$(1,0,1)$
T	J			
		N	$a_3\times(a_2\cdot(a_1\cdot I))$	$(1,1,0)$
	T	J		
		T	$a_3\cdot(a_2\cdot(a_1\cdot I))$	$(1,1,1)$

Note : The relays whose pickups are contacts A_2 and A_3 are multi-poled relays, that is, one pickup operates several relays.

Ohio State University
Columbus, Ohio

NOTE ON A FORMULA OF HERMITE

L. Carlitz

Hermite [2] proved that

(1)
$$\sum_{0 < r(p-1) < n} \binom{n}{r(p-1)} \equiv 0 \pmod{p},$$

where p is prime and n is odd. Bachmann [1, p. 46] showed that (1) holds for all n. Nielsen [3, p. 243] showed that (1) is implied by the Staudt-Clausen theorem for the Bernoulli numbers.

It may be of interest to consider the sum

(2)
$$S = \sum_{0 < rm < n} (-1)^{n-rm} \binom{n}{rm} \quad (p = 2m+1),$$

where p is an arbitrary odd prime.

We note first that (1) can be proved rapidly as follows. We have

$$\sum_{s=0}^{p-1} (s-1)^n = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \sum_{s=0}^{p-1} s^r \equiv -\sum_{r>0} (-1)^{n-r(p-1)} \binom{n}{r(p-1)},$$

where all congruences are (mod p) and we have used the familiar congruence $(n \geq 1)$

$$\sum_{s=0}^{p-1} s^n \equiv \begin{cases} -1 & (p-1 \mid n) \\ \\ 0 & (p-1 \nmid n). \end{cases}$$

But since

$$\sum_{s=0}^{p-1} (s-1)^n = \sum_{s=0}^{p-1} s^n,$$

(1) follows at once.

In the next place, the sum

$$S_2 = -\sum_{s=0}^{p-1} (s^2-1)^n = -\sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \sum_{s=0}^{p-1} s^{2r} = \sum_{r>0} (-1)^{n-rm} \binom{n}{rm}.$$

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Employing the Legendre symbol (s/p), we have also

$$\sum_{s=0}^{p-1} (s^2-1)^n \equiv \sum_{s=0}^{p-1} \left(1+\left(\tfrac{s}{p}\right)\right)(s-1)^n$$

$$\equiv \sum_{s=0}^{p-1} (s-1)^n + \sum_{s=0}^{p-1} \left(\tfrac{s}{p}\right)(s-1)^n$$

$$\equiv \sum_{s=0}^{p-1} s^n + \sum_{s=0}^{p-1} \left(\tfrac{s+1}{p}\right)s^n \ .$$

Now

$$\sum_{s=0}^{p-1} \left(\tfrac{s+1}{p}\right)s^n \equiv \sum_{s=0}^{p-1} (s+1)^m s^n$$

$$\equiv \sum_{r=0}^{m} \binom{m}{r} \sum_{s=0}^{p-1} s^{n+r}$$

$$\equiv \begin{cases} -\binom{m}{k} & (p-1\mid n+k) \\ \\ 0 & \text{(otherwise)} \end{cases}$$

Consequently

$$(4) \qquad S_2 \equiv \begin{cases} 2 & (p-1\mid n) \\ \binom{m}{k} & (p-1\mid n+k,\ 0<k\leq m) \\ 0 & \text{otherwise} \ . \end{cases}$$

Comparing (3) and (4) we get

$$(5) \qquad S \equiv \begin{cases} \binom{m}{k} & (p-1\mid n+k,\ 0\leq k\leq m) \\ 0 & \text{otherwise} \ , \end{cases}$$

where S is defined by (2).

Exactly as the congruence (1) is related to the Staudt-Clausen theorem for the Bernoulli numbers, so also (5) is related to an analogous theorem for the numbers

(6)
$$\Delta_n = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} D_r = 4^n \sum_{r=0}^{n} \binom{n}{r} B_{n+r} \,,$$

where [4, p. 28]

$$D_n = (2-2^n)B_n = (2B+1)^n \,.$$

The Staudt-Clausen theorem implies

$$pD_n \equiv \begin{cases} -1 & (p-1 \mid n) \\ 0 & (\text{otherwise}), \end{cases}$$

where p is an odd prime. Thus it follows from the first of (6) that

(7)
$$p\Delta_n \equiv -\sum_{r>0} (-1)^{n-rm} \binom{n}{rm} \equiv -S_2 \,.$$

Incidentally if we use the second of (6) we get

(8)
$$p\Delta_n \equiv -4^n \sum_{r \equiv -n \pmod{p-1}} \binom{n}{r} \,,$$

so that we have another sum related to S_2.

If $p = 3m + 1$ and we define

$$\left(\tfrac{a}{p}\right)_3 \equiv a^m \,,$$

then

$$\sum_{s=0}^{p-1} (s^3-1)^n = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \sum_{s=0}^{p-1} s^{3r}$$

$$\equiv -\sum_{r>0} (-1)^{n-rm} \binom{n}{rm} \,.$$

On the other hand

$$\sum_{s=0}^{p-1} (s^3-1)^n \equiv \sum_{s=0}^{p-1} \left\{ 1 + \left(\tfrac{s}{p}\right)_3 + \left(\tfrac{s}{p}\right)_3^2 \right\}(s-1)^n$$

$$= \sum_{s=0}^{p-1} s^n + \sum_{s=0}^{p-1} \{(s+1)^m + (s+1)^{2m}\} s^n$$

$$\equiv \sum_{s=0}^{p-1} s^n + \sum_{r=0}^{m} \binom{m}{r} \sum_{s=0}^{p-1} s^{n+r} + \sum_{r=0}^{2m} \binom{2m}{r} \sum_{s=0}^{p-1} s^{n+r}$$

Consequently

$$(9) \qquad S_3 = \sum_{r>0} (-1)^{n-rm} \binom{n}{rm} \equiv \begin{cases} 3 & (p-1 \mid n) \\ \binom{m}{k} + \binom{2m}{k} & (p-1 \mid n+k, \ k>0) \\ 0 & \text{otherwise .} \end{cases}$$

The corresponding Staudt-Clausen theorem is for the numbers

$$\Delta_n^{(3)} = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} D_{3r}$$

$$= (-1)^n \sum_{2r \leq n} \binom{n}{2r} D_{6r} .$$

Indeed

$$(10) \qquad p\Delta_n^{(3)} \equiv -S_3 \qquad (p=3m+1) .$$

For $p \equiv -1 \pmod 3$, $p > 3$, we get

$$(11) \qquad p\Delta_n^{(3)} \equiv -(-1)^n \sum_{r>0} \binom{n}{r}$$

$$\equiv \begin{cases} -1 & (p-1 \mid n) \\ 0 & \text{(otherwise) .} \end{cases}$$

As for $p = 3$,

$$3\Delta_n^{(3)} \equiv -(-1)^n \sum_{r>0} \binom{n}{2r}$$

$$\equiv (-1)^{n-1}(2^{n-1}-1)$$

$$\equiv \begin{cases} -1 & (n \text{ even}) \\ 0 & (n \text{ odd}) , \end{cases}$$

so that (11) holds for $p=3$ also. Note that the denominator of $\Delta_n^{(3)}$ is odd.

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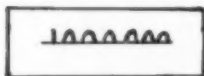
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LETTERS FROM SUBSCRIBERS

. . .

I am pleased to have discovered Mathematics Magazine. I particularly like your approach that mathematics, though serious, need not be dull. Perhaps that's because I am not a mathematician, but a collecter of the humerous and wry in mathematics. Such as

What mean all the mysteries to me
Whose life is full of indices and surds,

$$x^2 + 7x + 53 = 11/3$$

Lewis Carroll

or

      half a million

Maxey Brooks

. . . I have been told by several excellent mathematicians that they do not belong to the AMS nor any other mathematical society because many, many papers are meaningless to them because terms and notations are not defined. As one of the authors of the Mathematics Dictionary . . . you are certainly familiar with the various usage of numerous symbols, various interpretations of functions and, above all, the use of "vogue" terms to impress rather than to inform the reader. This last seems to be the "curse" of the intellectuals of today in many fields. It has grown so bad that often the "founder" of a new branch or twig of pure mathematics no longer can understand papers on it after a few years pass.

E. T. Bell's remark to the effect that one could not hope to become familiar with more than 5% of mathematics if he started young and worked at it long (in either his Men of Math. or The Development of Math.) would probably have to be revised downward today, partly from the growth of mathematics in the years since he wrote it but partly, also, from the plethora of terms and expressions for the same thing. Perhaps Math. Mag. should include a regular department of "clarification" to combat this trend.

Ben E. Dyer

A family of curves

ON THE USE OF THE EQUIVALENCE SYMBOL AND
PARENTHESES SYMBOLS IN ASSOCIATIVE
DISTRIBUTIVE ALGEBRA

H. S. Vandiver*

We have already treated the foundations of associative algebra, which include the foundations of the theory of integers, in four papers published in this magazine, each under the title "A Development of Associative Algebra and an Algebraic Theory of Numbers," appearing as follows:

(I) - Vandiver, vol. 25, 1952, pp. 233-250.

(II) - Vandiver, vol. 27, 1953, pp. 1-18.

(III) - Vandiver and Weaver, vol. 29, 1956, pp. 135-149.

(IV) - Vandiver and Weaver, vol. 30, 1956, pp. 1-8; Errata, vol. 30, 1957, p. 219.

Part of this paper may, in effect, be regarded as a supplement to one or more of these papers. In fact, we shall pursue much further some of the ideas expressed in the above papers concerning use of the equivalence symbols (= or ~), and also the use of parentheses.

In connection with the first topic mentioned in the title, we note that two advances have been made during the history of mathematics that some mathematicians refer to as the greatest advances made, in that period, in mathematical thought, if not all abstract thought. These are, first, the recognition by the ancient Greeks, in particular Thales, of the desirability of beginning the discussion of geometry by setting up a system of postulates, that is, postulates or axioms presumably self-evident. The second step of this character was made by Lobachewsky, who, in his invention of non-Euclidean geometry, in effect discarded the notion that postulates or axioms should be self-evident. We think it is quite possible that historians and critics of mathematics in the future may refer to the development of the idea that we can, for example, use algebraic formulas and transform them according to a set of axioms which do not depend on defining any particular symbol in itself, but only on the manipulation of the symbols in the formulas. *When we follow this scheme we seem to be getting closer and closer to Hilbert's idea that mathematics is the art of manipulating meaningless symbols.* Related to this is the fact that algebraists at the present time are particularly active in reducing other parts of mathematics to algebraic patterns, such as the subject which Artin calls geometric algebra.

It seems to me that in spite of the fact that considerable advances

have been made along the lines just mentioned, there is one symbol which always, or nearly always, in mathematical discussions has had a direct relation to logic, as any formula is usually regarded as a logical statement, and it is usually said that the expression "$4+2 = 6$" means "$4+2$ is 6." In view of the latter relation we should then be able to say that $4+2$ is $5+1$ is $3+2+1$, etc. However, the number theorist in his consideration of such relations does not necessarily regard $4+2$ as being 6. For example, in considering partitions, he generally states that $4+2$ is a partition of 6 or that $4+2$ and $5+1$ are *different* partitions of 6. It has always seemed to us that these two points of view are quite at variance. In view of this situation we shall try to divest, at least, the symbol "$=$" from any relation to logic, *after we describe how we shall manipulate our symbols*. In this connection we shall quote R. L. Wilder [1] concerning the ideas of Poincaré:

> *Poincaré rejected and ridiculed attempts to base mathematics on logic.* He asked, "If ... all the propositions [which mathematics] enunciates can be deduced one from another by the rules of formal logic, why is not mathematics reduced to an immense tautology? The syllogism can teach us nothing essentially new, and, if everything is to spring from the principle of identity, everything should be capable of being reduced to it. Shall we then admit that the enunciations of all those theorems which fill so many volumes are nothing but devious ways of saying A is A?"

It is possible that Poincaré had in mind some such difficulties as we just described concerning "$4+2$ is 6."

Connected with our attempts to go as far as possible in obtaining formulas without using the ideas of logic, for many years the writer considered the problem of describing *all* the operations which mathematicians go through in manipulating formulas in elementary algebra. To his great surprise he was never able to find that any previous writer had given rules or axioms to cover the handling of parentheses in such formulas, so an attempt was made in 1952 (cf. article I, pp. 241-2) to set up a system of postulates which would enable one to carry out any or *all* of the usual steps in an algebraic argument which involves parentheses. Related to this lack in the previous literature was the apparent fact that no one had given complete rules for making valid substitutions in algebraic equations, and the axioms just mentioned covered such operations also.

In the present paper we shall discuss the use of parentheses further, particularly in connection with the possibility of writing each algebraic formula without the use of parentheses as was done by Lukasiewicz,[7] but we shall first describe the system of symbols and the axioms we use. This has been done in our article I, but as the contents of the latter do not seem to be well-known, we shall repeat it here.

Starting with the natural numbers, we postulate that 1 is a natural number and also that each natural number has an immediate successor in a set of natural numbers. We also indicate that 1 is not the immediate

successor of any natural number, or we may say that 1 is the first term in the sequence of natural numbers. We do not use any other of the Peano postulates except that of induction. Consider the symbols

(1) $$C_1, C_2, C_3 \dots$$

where each subscript is a natural number, and the immediate successor of C_k where k is a natural number is $C_{k'}$ where k' is the immediate successor of k in a set of natural numbers. We introduced (Article I, p. 242) in addition to these symbols a symbol + (called a plus sign), and × (called a multiplication sign), and ((called a left parenthesis symbol), and) (called a right parenthesis symbol). We now define the term *combination* in connection with the symbols. Two definitions of this term *combination* will be given; the first is as follows:

> Any of the symbols in (1) or any symbol denoting any of them is said to be a *combination*. If A denotes a combination and B also, then $A + B$ is said to be a *combination*, also A, (A), and $A \times B$. (So far in this work, we have encountered no difficulty in using the idea that a symbol may denote itself.)
>
> A *sub-combination* of a combination A is a combination consisting of a symbol contained in A or else such a symbol followed by others in order as they appear in A.

The second definition is as follows: (Article I, p. 249, fn. 9):

> Consider a *finite linearly ordered set* (or *sequence*) of symbols containing only symbols of the following type: symbols (letters) denoting elements in a set of symbols described in (1), symbols of conjunction + and ×, *parenthesis symbols* (and) which will be called a *left parenthesis symbol* (abbreviated L. P. S.) and a *right parenthesis symbol* (abbreviated R. P. S.), respectively, and such that:
>
> 1. It contains at least one symbol denoting an element of (1).
> 2. It begins with either an L. P. S. or a symbol denoting an element of (1) and ends with either an R. P. S. or a symbol denoting an element of (1).
> 3. It has no L. P. S. immediately preceding a symbol other than another L. P. S. or a symbol denoting an element of (1); and no R. P. S. or a symbol denoting an element of (1).
> 4. Any two successive symbols denoting elements of (1) are separated by just one symbol of conjunction.
> 5. The instances of the symbols (and) can be paired into sensed [2] pairs (,).

Definition: If A denotes a combination, then (A) is called *parenthesis enclosed combination*.

Definition: A *closed combination* C is a combination such that if any + sign occurs in it, there is a sub-combination of C which contains this + sign, and which is also a parenthesis enclosed combination. If a combination contains no plus sign it is said to be closed.

We now introduce a symbol of relation between combinations, $=$, called equality. *However, we do not yet employ the idea of inequality. In particular we make no statements at this stage as to the elements in* (1) *being* equal or not equal. Using the equality symbol we shall now set up six postulates (Article I, p. 243) involving the symbols C in (1). In the following statements each capital letter, or capital letter primed, denotes an arbitrary combination as above described or a combination limited in character by the conditions in the statements. A small letter denotes an arbitrary natural number, or else a natural number limited in character by the conditions in the statement. An equality is also called a statement, but in our discussion, the latter word does not necessarily mean "logical statement."

Postulate 1. (Identity) $A = A$.

Postulate 2. (Parenthesis) $(A) = A$.

Postulate 3. (Substitution)[3] If $A = B$ and $D = C$, where C denotes a sub-combination of B and B' denotes the combination obtained from B by putting D in place of C, then $B' = A$, provided that if C is immediately preceded by or immediately succeeded by an $\times$ sign in B, then C and also D must be closed combinations.

Postulate 4. (Induction) For each natural number n let there be associated a statement denoted by $S(n)$. If $S(1)$ holds, and if it follows that if $S(a)$ holds, then $S(a')$ holds, where a' is the immediate successor of a in the set of natural numbers, then $S(n)$ holds for each natural number n.

Postulate 5. (Addition) If n denotes a natural number and n' denotes the immediate successor to this number in the set of natural numbers, then

$$C_n + C_1 = C_{n'}.$$

Postulate 6. (Multiplication)

$$C_a \times (C_b + C_1) = C_a \times C_b + C_a$$

$$C_a \times C_1 = C_a.$$

The use of the above postulates enables us to prove the following properties of the C's: symmetry involving the C's and their combinations, transitivity, composition under addition and under multiplication, general substitution, commutative laws of addition and multiplication, the closure laws of addition and multiplication, and the distributive law.

Vandiver and Weaver (Cf. (IV), pp. 3-8) treated an algebraic system which they called a semiring. This system has the properties which follow: We shall use the equivalence symbol $(=)$, the additive symbol $(+)$, and the multiplication symbol $(\times)$ (which sign is usually omitted), in addition to the usual parentheses signs. We may then define combinations in a way similar to the manner in which combinations were defined with respect to (1) of the present paper. Then we may state the postulates:

    (i) G is a semigroup relative to $+$ and $=$.

    (ii) G is a semigroup relative to $\times$ and $=$.

(iii) Whenever S_1, S_2, $S_3 \in G$, then $S_1(S_2 + S_3) = S_1 S_2 + S_1 S_3$ and $(S_2 + S_3)S_1 = S_2 S_1 + S_3 S_1$.

(iv) Substitution, as in our treatment of the C's in (1) of the present paper, holds in G,

In view of the above and our statements concerning the theorems derived from our postulates which follow their statement, it will be found that the C's in (1) form a semiring.

We shall now show how, by making assumptions as to equality among the C's in (1), we may obtain various types of semirings. First, let us suppose that if $C_i = C_j$, then both i and j denote the same natural number. In other words, we can then define inequality among this particular set of C's by saying that if C_m and C_k with m and k denoting different natural numbers, the C_k and C_m are said to be unequal, and we use the symbol $C_m \neq C_k$. Hence the set of C's is *isomorphic*[4] with the set of natural numbers themselves. *This idea of inequality is introduced here for the first time. Because of this isomorphism just mentioned we have developed some of the fundamental properties of the natural numbers without employing the law of excluded middle in the sense that we have not used anywhere the statement $C_a = C_b$, $C_a \neq C_b$, are mutually excluded. Hence the theorems we have proved concerning the C's will establish the result that*

(2)
$$C_4 + C_3 = C_7,$$

and consequently $4 + 3 = 7$ without any idea of possible contradiction being introduced. Hence (2) may be regarded as merely a linearly ordered set of symbols, or a sequence of symbols, without any reference to any ideas with regard to logic in the usual sense. Of course a similar statement can be made about what is probably a majority of known results in elementary algebra, the exceptions being those that seem to require an indirect proof.

The treatment of natural numbers, described above, in effect *avoids the use of Peano's axiom which usually reads as follows:* If a and b denote positive integers with a' denoting the immediate successor of a and b' denoting the immediate successor of b, then if $a' = b'$ then $a = b$. In our treatment, due to the fact that the equivalent of this axiom was omitted, in order to give a proof of this statement, the indirect method appears necessary.

The idea just expressed may be brought out in a simpler way by considering a denumerable set of symbols which form a semigroup $\mathbb{S}$. Suppose $e_1 \in \mathbb{S}$ and $e_2 \in \mathbb{S}$, and further suppose that

(3)
$$e_1 a = a e_1 = a$$

holds for any $a \in \mathbb{S}$, and similar relations hold when e_1 is replaced by e_2 in (3). Further, we agree not to use any assumption concerning inequality among the elements of the semigroup.

We shall now show that $e_1 = e_2$ in (3). By the assumptions concerning the latter relation we have $e_1 e_2 = e_2$ and also $e_1 e_2 = e_1$, whence $e_1 = e_2$, which is the result. *Now if we talk about a possible contradiction being involved in this argument this must necessarily be of the form that c = d and c ≠ d simultaneously with c ε Ⓢ, d ε Ⓢ; however, we had agreed not to employ the idea of inequality involving the elements of Ⓢ.*

We shall now discuss the way ordinary logic may enter into manipulation of an equation involving natural numbers, using originally the axioms employed here for the C's, and further for the set of natural numbers when it is isomorphic to them. The equation referred to is a finite sequence of symbols, and in describing how we intend to employ them in the order of the sequence it might be convenient to say, for example, a follows b or b follows a, and that these two statements are mutually exclusive.[5]

The other characteristic of logic which usually takes the form "if A holds, then B follows," may be avoided in a large part of algebra if we are enabled, in view of our axioms, to regard each equation as a sequence of symbols. Then we may describe our usual manipulation of equations by saying that we are obtaining in succession a number of sequences of symbols by the removal of some symbols, changing the order of the symbols in the sequence, and introducing new symbols in one or more of our sequences. Further, we might call any one of such sequences a super-combination, as a generalization of our original idea of combination, each super-combination containing the symbol "=" just once. Also, we could call $3 + 4 = 7$ a *preferred* super-combination and could call $3 + 4 = 5$ *non-preferred.*[6]

We have already noted that when we defined the set (1) no assumption was made concerning the equality or inequality of the C's. We are then, for example, ready to assume that

$$(4) \qquad\qquad C_7 = C_3 ,$$

and then we see immediately, in view of postulate (5), that we have only a finite set of unequal elements. In particular, from the same postulate,

$$C_6 + C_1 = C_2 + C_1 .$$

Yet we cannot cancel the C_1's if we apply the assumption to this set of elements (already referred to in the case when all C's were distinct) that any two C's cannot be equal and unequal at the same time. This semiring has no element having the property of a zero element. In view of (4), the elements in (1) in the sense of equality, repeat in cycles the elements in each cycle equaling C_3, C_4, C_5, and C_6. Under addition, these elements form a cyclic group. The above described semiring is a special case of the semiring formed by the set

$$(5) \qquad C_1 C_2 \cdots C_{j-1}; \quad C_j = C_i; \quad i > j; \quad j > 1,$$

and the C's distinct. Largely because of the fact that the cancellation

law of addition does not hold in this algebra, and the possibility that the idea was new in the year 1934 when the writer's first paper on it was published, I imagine because of the history of similar matters that some mathematicians may be quite allergic to these ideas. However, to any mathematician who has worked in the theory of semigroups, this will probably not be the case since perhaps the simplest type of semigroup is the finite cyclic semigroup written usually in the *multiplicative* form as

(6)
$$A, A^2, \cdots, A^{j-1},$$

with $A_j = A_i$ for some $i < j$, with the elements of (5) distinct and $j > 1$, and this is isomorphic to the set (5) under *addition*.

The reader may now ask what advantage is gained to mathematicians by these efforts to make the manipulation of algebraic equations as "meaningless" as possible. As a matter of fact this has resulted in quite considerable generalizations of the notion of equality or equivalence, examples of which we shall give in what follows :

Let us again consider the natural numbers and introduce a relation between them called (i, j) equivalence where $j \geq i$. We use the symbol $\equiv$ and define it as follows (each letter denoting a natural number): If $a < i$, then $a \equiv b$, if and only if $a = b$; if $a \geq i$ and $b \geq i$; then $a \equiv b$ if and only if $a \equiv b \pmod{m}$, where $m = j - i$. Addition and multiplication yield the same elements as in ordinary arithmetic, except that in an (i, j) algebra we may "reduce" elements greater than $j - 1$.

In view of the above there are exactly $j - 1$ natural numbers which are not (i, j) equivalent, namely 1, 2, $\cdots$, $j - 1$. (This idea of (i, j) equivalence is due to A. Church (cf. our Article IV, pp. 5-6.) It is clear that the above system is isomorphic to the set (5) involving the C's, where C_a in the set (5) maps on the natural number a under (i, j) equivalence. Let us now consider the special case when $i = 1$ in (5). Then this set is isomorphic to the set

$$1, 2, \cdots, j - 1,$$

modulo $n = \overline{j - 1}$. It is evident, then, when $i = 1$, that the symbol $\equiv$ may be replaced by the ordinary symbol $=$. Each of these two systems is isomorphic to the set of residue classes modulo n. Here we have, then, three systems which are all isomorphic, and we were led to them, in effect, by using our equivalence symbol in three different ways.

We shall now discuss the use of parentheses in connection with the C's *in* (1). This is covered by our definition of "combination" and postulates 4 and 5. However, as Lukasiewicz[7] has shown, *it is possible to express any combination without using parentheses at all, as was pointed* out to me recently by Karl Menger. This may be illustrated as follows : What we usually write as $a(b + c)$ could be written as $PaSbc$ where P stands for product and S for sum. Also, what we write as $(c \cdot a) + b$ could be written as $SPcab$. This way of writing combinations is quite valuable from the standpoint of illustrating their innate simplicity, and *it is closely*

related to the first definition of combination which we have given in this paper where parentheses were used. (Cf. Article I, p. 242.) However, even if mathematicians felt that it might be a good idea to employ this Lukasiewics notation, and wrote all their original papers in the future using it, it would take a great deal of time for the investigator to translate the papers, written in the past, employing algebraic formulation into the new notation in order to follow the material in these older papers. Also, for example, in the use of a non-parenthesis notation he might solve a system of linear algebraic equations in several unknowns and obtain, after a number of transformations, an expression of the form

$$h(3a+(b+c))((k+l)+s)+t,$$

where all the letters denote known quantities. It might be quite difficult to carry out all the operations in the elimination and employ at each step the proposed new notation.

FOOTNOTES

*The author's work on this paper was done under Basic Research Grant 8238, which was awarded to him by the National Science Foundation.

1. *Introduction to the Foundations of Mathematics,* John Wiley and Sons, New York, p. 202.

2. This statement appears to be equivalent to what was given as (5) in our original definition. (Cf. Article I, p. 249, bottom of page.)

3. Concerning our substitution postulate (3), the nearest previous approach to this that we have so far noted is given by G. D. Birkhoff and R. Beatley (*Basic Geometry,* Scott, Foresman and Company, New York, 1933, p. 285). It reads as follows:

     4. If $x = y$, then $f(x, a, b, c, ...) = f(y, a, b, d, ...)$, where the expression $f(x, a, b, c, ...)$ denotes a real number built up from successive combinations of the numbers x, a, b, c, ... and the operations + and ×, and $f(y, a, b, c, ...)$ denotes the number obtained from $f(x, a, b, c, ...)$ by writing y in place of x throughout.

   If we confine ourselves to the natural numbers in the above statement, then our substitution postulate referred to goes further in that the use of parentheses is described and the order of the elements in the combinations is defined in full.

4. In algebra, at least, one of the most valuable tools is *homomorphism.* This seems to be a concept which is not employed in logic. *Related to this is the idea that perhaps we should regard logic as a particular branch of mathematics, instead of the reverse.*

5. Cf. for example, our definition of *combination* in this paper.

6. Some mathematicians hold that it is impossible to separate logic and mathematics. In this paper we are indicating that it seems possible to do this in a limited fashion, but when we use language in connection with anything whatever, it could be that the user is employing the law of excluded middle instinctively in many of his verbal statements. So when we reach a point in our discussions of formulas when we wish to give "meaning" to some of the symbols and use language to indicate some of this meaning, we probably are using logic in some form.

7. Paul C. Rosenbloom, *The Elements of Mathematical Logic,* Dover Publications,

SOME INTEGRALS FOR GENOCCHI NUMBERS

J. M. Gandhi

1. Introduction. The Genocchi numbers G_N are defined by the generating function

$$\frac{2x}{(e^x+1)} = \sum_{N=1}^{\infty} G_N \frac{x^N}{N!} \quad . \quad \text{(Refs. 1. & 2.)} \tag{1.1}$$

We can also define them by the formulae

$$G_N = -2(2^N-1)B_N \tag{1.2}$$

$$C_{N-1} = 2^{N-1} \frac{G_N}{N} \tag{1.3}$$

Where B_N are the Bernoulli's numbers and the notation C_N (Tangent-Co-efficients) is due to Nörlund. (Ref. 3.)

The Integral

$$\int_0^{\infty} x^{2N-1} \operatorname{cosech}(\pi x)\, dx = (-)^N \frac{G_{2N}}{4N} \tag{1.4}$$

is the known integral for Genocchi numbers. (Ref. 4) In this note we shall prove the following integrals for these numbers.

$$\int_0^{\infty} x^{2N-2} \log(\coth \pi x/2)\, dx = (-)^N \frac{G_{2N}\pi}{2 \cdot 2N \cdot (2N-1)} \tag{1.5}$$

$$\int_0^{\infty} x^{2N} \coth(\pi x) \operatorname{cosech}(\pi x)\, dx = (-)^N \frac{G_{2N}}{2\pi} \tag{1.6}$$

2. Now

$$\int_0^{\infty} x^{2N-2} \log(\coth \pi x/2)\, dx = \int_0^{\infty} x^{2N-2} \log\left[\frac{e^{\pi x/2}+e^{-\pi x/2}}{e^{\pi x/2}-e^{-\pi x/2}}\right] dx$$

$$= \int_0^{\infty} x^{2N-2} \log\left[\frac{1+e^{-\pi x}}{1-e^{-\pi x}}\right] dx$$

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$$= \int_0^\infty x^{2N-2} \{ \log(1+e^{-\pi x}) - \log(1-e^{-\pi x}) \} \, dx$$

$$= 2 \int_0^\infty x^{2N-2} (e^{-\pi x} + e^{-3\pi x}/3 + e^{-5\pi x}/5 + \cdots) \, dx \tag{2.1}$$

Since it can be proved that

$$\int_0^\infty x^{2N-2} e^{-k\pi x} \, dx = \frac{(2N-2)!}{k^{2N-1} \pi^{2N-1}} \tag{2.2}$$

Right hand side of (2.1) becomes

$$\frac{2(2N-2)!}{\pi^{2N-1}} (1/1^{2N} + 1/3^{2N} + 1/5^{2N} + \cdots) \tag{2.3}$$

Now we know that

$$1 + 1/3^{2N} + 1/5^{2N} + \cdots = (-)^{N-1} \frac{(2^{2N}-1)B_{2N}\pi^{2N}}{2(2N)!}$$

(Ref. 7. Equ. 137. Page 238)

$$= (-)^N \frac{G_{2N}\pi^{2N}}{4\cdot(2N)!} \tag{2.4}$$

Using the results 2.3, 2.4 from 2.1 we get

$$\int_0^\infty x^{2N-2} \log(\coth \pi x/2) \, dx = (-)^N \frac{G_{2N}\pi}{2\cdot 2N \cdot (2N-1)} \tag{2.5}$$

Now consider

$$\int_0^\infty x^{2N} \coth(\pi x) \operatorname{cosech}(\pi x) \, dx = 2 \int_0^\infty x^{2N} \frac{(e^{\pi x}+e^{-\pi x})}{(e^{\pi x}-e^{-\pi x})^2} \, dx$$

$$= 2 \int_0^\infty x^{2N} e^{-\pi x} \frac{(1+e^{-2\pi x})}{(1-e^{-2\pi x})^2} \, dx \tag{2.6}$$

Since

$$\frac{(1+t)}{(1-t)^2} = 1 + 3t + 5t^2 + 7t^3 + \cdots \tag{2.7}$$

In view of 2.7 R. H. S. of 2.6 becomes

$$2\int_0^\infty x^{2N}(e^{-\pi x}+3e^{-3\pi x}+5e^{-5\pi x}+\cdots)\,dx = \frac{2(2N)!(1/1^{2N}+1/3^{2N}+1/5^{2N}+\cdots)}{\pi^{2N+1}}$$

$$= 2(2N)!(-)^N\frac{G_{2N}\pi^{2N}}{4\cdot(2N)!\pi^{2N+1}}$$

(In View of 2.4)

$$= (-)^N\frac{G_{2N}}{2\pi} \tag{2.8}$$

Thereby we have proved 1.6.

For other integrals for Bernoulli's and related numbers see Refs. 5. & 6.

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Dept. of Physics
Thapar Polytechnic
Patiala (E. Punjab)
India

A NOTE ON MATHEMATICAL NOTATION

H. A. Pogorzelski

In the last few years mathematics has been tending toward extensive use of superscripts and subscripts in its notation. Such usual "first-order" superscripts and subscripts as in B^n, B_m give mathematicians and editors and compositors no difficulties. However, serious difficulties arise with respect to "n-th order" superscripts and subscripts. Matter-of-fact, mathematicians are unfortunately limited to 2-nd order superscripts and subscripts, for example as in B^{n_k}, B_{m_k}, primarily because of the printing difficulties and costs involved with superscripts and subscripts of orders greater than 2. It follows that a notational extension is needed. It is therefore suggested here that the following notations of Peano and the author be considered. (In justice to MATHEMATICS MAGAZINE, I confine my examples to the 3rd order superscripts and subscripts. The n-th order case will be obvious.)

For the symbol "$\vdash$" in "$\vdash n$" read "superscript n", for the symbol "$\dashv$" in "$\dashv m$" read "subscript m", and the symbol "$\updownarrow$" in "$m\updownarrow n$" read "m subscript and n superscript". The following examples illustrate the notation on the seven possibilities that occur with superscripts and subscripts of order 3 :

$$B^{n^{k_2}} = B^{n \vdash k \vdash 2}, \quad B^{n^{k^2}} = B^{n \vdash k \vdash 2},$$

$$B_{m_{k_2}} = B_{m \dashv k \dashv 2}, \quad B_{m_{k^2}} = B_{m \dashv k \vdash 2},$$

$$B_{m_{k_2}}^{n_{k_2}} = B_{m \dashv k \dashv 2}^{n \vdash k \vdash 2} \quad \text{or} \quad B_{m \dashv k \dashv 2 \updownarrow n \vdash k \vdash 2},$$

$$B_{m^{k^2}}^{n^{k^2}} = B_{m \vdash k \vdash 2}^{n \vdash k \vdash 2} \quad \text{or} \quad B_{m \vdash k \vdash 2 \updownarrow n \vdash k \vdash 2},$$

$$B_{m^{k^2}}^{n^{k_2}} = B_{m \vdash k^2 \updownarrow n \vdash k_2}.$$

It is important to note that anything appearing left of "$\updownarrow$" is always the subscript, and anything on the right of "$\updownarrow$" is the superscript. Finally, we give a simple example of the n-th order superscript :

$$B^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}} = B^{\omega \vdash \cdots \vdash \omega}.$$

Other more special uses of this notation will appear in "The American Mathematical Monthly".

Henry Pogorzelski
American Mathematical Society

TEACHING OF MATHEMATICS

Edited by

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BISECTORS OF SUPPLEMENTARY ANGLES

A NEW LOOK

R. Lariviere

If the arms of a pair of supplementary angles are $L_1 = 0$, $L_2 = 0$, the bisectors of the angles are members of the set of lines $L_1 + kL_2 = 0$. Since the product $L_1 L_2 = 0$ is a degenerate hyperbola the bisectors are also the transverse and conjugate axes of this hyperbola.

The equation $L_1 L_2 = 0$ may be written

$$(1) \qquad Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \ .$$

Hence, if $B \neq 0$, rotation of the coordinate axes through an angle θ given by

$$(2) \qquad \cot 2\theta = (A - C)/B$$

would eliminate the xy term in (1) by setting the coordinate axes parallel to the axes of the hyperbola (the bisectors). Since $\cot 2\theta = (1 - \tan^2\theta)/(2\tan\theta)$, equation (2) yields two values m_1, m_2 for $m = \tan\theta$. If rotation through $\arctan m_1$ would set the x axis parallel to the transverse axis (the first bisector), then rotation through $\arctan m_2$ would set the x axis parallel to the conjugate axis (the second bisector). Thus m_1 and m_2 are the slopes of the bisectors. If m_1 and m_2 are equated successively to the slope of $L_1 + kL_2 = 0$, corresponding values k_1, k_2 are determined. Then $L_1 + k_1 L_2 = 0$ and $L_1 + k_2 L_2 = 0$ are the equations of the bisectors.

If $B = 0$ in (1), the axes of the hyperbola are already parallel to the coordinate axes. That is, the bisectors are lines $x = a$, $y = b$. To obtain the corresponding values of k_1 and k_2, the coefficients of the two varia-bles in $L_1 + kL_2 = 0$ are successively equated to zero. Then again $L_1 + k_1 L_2 = 0$ and $L_1 + k_2 L_2 = 0$ are the equations of the bisectors.

Example: To find the equations of the bisectors of the supplementary

angles formed by $3x + 4y = 20$ and $5x - 12y = 26$, we take $(3x + 4y - 20)(5x - 12y - 26) = 0$, that is, $15x^2 - 16xy - 48y^2 + \cdots = 0$. Then $(15 + 48)/(-16) = (1 - \tan^2\theta)/(2\tan\theta)$, and $\tan\theta = 8$ or $\tan\theta = (-1/8)$. All the lines $(3 + 5k)x + (4 - 12k)y - (20 + 26k) = 0$ through the vertex of the supplementary angles have slopes $-(3 + 5k)/(4 - 12k)$. When $m = 8$, $k = 5/13$, and when $m = (-1/8)$, $k = -5/13$. Consequently, $32x - 4y - 195 = 0$ is the equation of the bisector whose slope is 8, and $7x + 56y - 65 = 0$ is that of the bisector whose slope is $(-1/8)$.

University of Ill.
Navy Pier, Chicago 11

A PRIMITIVE GEOMETRY

Curtis M. Fulton

In this paper we exhibit some properties of a geometry which may be called "primitive" for at least two reasons : first, the solution of triangles requires only one formula and second, a general transformation of coordinates may be written involving only one arbitrary constant.

Our geometry will be based on axioms closely resembling those for vector spaces. The scalars to be used are denoted by capital letters and are consistently thought of as real numbers. Now we consider a set of elements, denoted by small letters, satisfying axioms to be specified. Any two elements x and y determine a unique element $x+y$, called their sum. Also, to every scalar A and every element x there corresponds a unique element Ax as product. Addition and scalar multiplication are governed by the usual axioms for vector spaces [2, pp. 3-4]. To prevent misunderstandings we use the letter θ for the null element. The notation (x, y) indicates the inner product of the ordered pair of elements x and y. It is a real number such that

(1) $$(x+y, z) = (x, z) + (y, z), \quad (Ax, y) = A(x, y);$$

(2) $$(x, y) = -(y, x);$$

(3) $\quad (x, y) \neq 0 \quad$ if x, y are linearly independent.

The axioms just stated are identical to those used in [1]; most of the geometric interpretations, however, will be quite different.

One of our tools is the fundamental identity for any three elements

(4) $$(y, z)x + (z, x)y + (x, y)z = \theta$$

which was shown in [1] as a direct consequence of the axioms. Furthermore, a new symbol, namely (xyz), is defined by

(5) $$(xyz) = (y, z) + (z, x) + (x, y).$$

Straightforward use of (4) and (5) yields the new identity

(6) $$x(g, y-z) + y(g, z-x) + z(g, x-y) = g(xyz).$$

In order to get a *primitive geometry* out of our set of elements we need a number of meaningful geometric definitions. First of all the elements will be called *points*. Next we define $(g, x-y)$ for a given fixed element $g \neq \theta$ as the directed *distance* between the points x and y. We may speak of two distinct points as *parallel points* if their distance vanishes. Three non-parallel points are defined to be *collinear* if and only if

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(7) $x(g, y-z) + y(g, z-x) + z(g, x-y) = \theta$.

Hence it follows from (6) that $(xyz) = 0$ is a necessary and sufficient condition for three points to be collinear.

Since relation (7) is equivalent to $(x-y)/(g, x-y) = (x-z)/(g, x-z)$ it expresses the fact that

(8) $w = (x-y)/(g, x-y)$

is the same for all pairs of points on the *line*. It is therefore proper to let w represent the *direction* of the line. If the line through x and y and the line through p and q have the same direction it is not hard to see that

$$0 = (x-y, p-q) = (xpq) - (ypq).$$

Thus, if the lines have one point in common, they are identical. Two distinct lines with the same direction have no common points and are called *parallel lines*. On the other hand, assuming that the directions of the above lines are different consider the point m satisfying

$$(x-y, p-q)m = -(ypq)x + (xpq)y.$$

This point is easily proven to lie on both lines. Finally, the inner product of two directions is taken to be the directed *angle* of the lines involved. Hence two parallel lines determine a zero angle.

Take three non-collinear points, no two parallel, x, y, z. They are the vertices of a *triangle* where the notations u, v, w apply to the directions of the opposite sides. With the aid of (8) we find

(9) $(u, v)(g, y-z)(g, z-x) = (xyz)$.

From this we derive immediately

$$(u, v)/(g, x-y) = (v, w)/(g, y-z) = (w, u)/(g, z-x)$$

which is the only formula needed for the solution of triangles, since the directed sides and also the angles add up to zero. Clearly, there will be two kinds of similar triangles, corresponding angles or corresponding sides being equal, respectively. More generally, we have *duality* of point and line as well as distance and angle.

The *area of a triangle* may be defined as one-half of the expression (9). This definition satisfies the usual requirements. Certainly *congruent* triangles have equal areas. Also, if we partition the triangle by means of a transversal it becomes obvious that its area is equal to the sum of the areas of its constituent triangles.

A *coordinate system* may be based on two fixed points f, g such that $(f, g) = 1$ and g is the point previously used. We choose $X_1 = (x, g)$ and $X_2 = (f, x)$ as coordinates of a point x and infer from (4) that $x = X_1 f + X_2 g$. It follows that distance is expressed by $(g, x-y) = Y_1 - X_1$. We also observe that a *transformation of coordinates* must leave the element g

occurring in the definition of distance unchanged. Again, the new base point f' is a linear combination of f and g. Then $X_2' = (f', x)$ is linear and homogeneous in X_1 and X_2. Furthermore the transformation should leave (9) invariant because of its geometric meaning. As a special case

$$(xyg) = X_1 Y_2 - X_2 Y_1 + Y_1 - X_1$$

should remain unchanged. Thus necessarily a change of coordinates is given by equations of the type

$$X_1' = X_1, \quad X_2' = A X_1 + X_2$$

with one parameter A.

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University of California at Davis

APPLICATIONS OF UNITY RATIOS

J. G. Luter

If both members of an equation are divided by one of the members, the quotients have unity value. Thus, if $A = B$, then $A/B = 1$ and $1 = B/A$. From the formula for the area of a triangle, $A = \frac{1}{2}bh$, we obtain the unity ratios $2A/bh$ and $bh/2A$. From the equation of equivalent denominate quantities, $1\,\text{ft.} = 12\,\text{in.}$, we obtain the unity ratios $1\,\text{ft.}/12\,\text{in.}$ and $12\,\text{in.}/1\,\text{ft.}$

The foregoing examples indicate that from any equation we may obtain two unity ratios. It is apparent that the product of any quantity and a unity ratio is equal to the original quantity. Thus, if $A = B$, then $6 = 6A/B$.

A familiar formula for the area of a rectangle is $A = LW$. Suppose the following relationships are given:

$$L = 6a, \quad W = 4b, \quad a = \frac{1}{2}s \quad \text{and} \quad b = \frac{1}{3}s.$$

Suppose it is desired to express A in terms of s. We might use the technique of substituting quantities for their equals. Substituting $6a$ for L, and $4b$ for W, we obtain $A = 6a(4b) = 24ab$. Substituting $\frac{1}{2}s$ for a, and $\frac{1}{3}s$ for b, we obtain

$$A = 24\left(\frac{s}{2}\right)\left(\frac{s}{3}\right) = 4s^2$$

The foregoing result may be obtained by applying unity ratios as follows:

$$A = LW\left(\frac{6a}{L}\right)\left(\frac{4b}{W}\right)\left(\frac{s}{2a}\right)\left(\frac{s}{3b}\right) = 4s^2$$
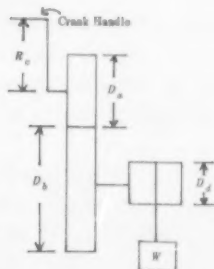
The following problem will illustrate more applications of unity ratios. Given: A hoist mechanism as shown in the sketch below.

Let F represent the effective force applied to the crank handle.

Let W represent the weight attached to the cable suspended from the drum.

Let E represent the efficiency of the hoist.

Required: Derive a formula to express the magnitude of the weight, W, that can be lifted when a force, F, is applied to the crank handle. Express W in terms of F, E, and other quantities if needed, provided that the other quantities are confined to the dimensions indicated in the sketch.

To aid in the derivation, we note the following relationships that are fixed by the mechanical arrangement of the parts.

1. The speed, V_h, of the crank handle equals $2\pi R_c N_c$ in which N_c represents the rotational speed of the crank.

2. The rotational speed, N_a, of wheel "a" equals N_c.

3. The peripheral speed, V_a, of wheel "a" equals $\pi D_a N_a$.

4. The peripheral speed, V_b, of wheel "b" equals V_a and also $V_b = \pi D_b N_b$ in which N_b represents the rotational speed of wheel "b".

5. The rotational speed, N_d, of the drum equals N_b.

6. The peripheral speed, V_d, of the drum equals $\pi D_d N_d$.

7. The speed, V_w, at which weight moves equals V_d.

To derive the desired formula we note the definition of efficiency:

$$E = \frac{\text{Power output}}{\text{Power input}} = \frac{\text{Output force (output speed)}}{\text{Input force (input speed)}}$$

When the weight is being raised, the output force is W and the input force is F; hence, $E = WV_w/FV_h$. Solving for W, we obtain $W = EFV_h/V_w$.

We note that the quantities V_h and V_w are not to appear in the final formula. These quantities must be eliminated from the above equation, but all changes in the form of the equation must be accomplished by valid mathematical procedures.

It would be possible to eliminate V_h from the equation by substituting the equivalent $2\pi R_c N_c$. This substitution would introduce into the equation the dimension, R_c, which is acceptable. It would also introduce the rotational speed, N_c, which is not acceptable. N_c could be eliminated by another substitution. If the substitution used to eliminate N_c should introduce another unacceptable quantity, it might be eliminated by another substitution. Such a procedure produces a series of changes and involves rewriting the equation several times. Instead of making substitutions, let us employ unity ratios.

To the right member of the equation $W = EFV_h/V_w$, let us apply as a factor the unity ratio $2\pi R_c N_c/V_h$, so as to cancel V_h. Then let us apply the unity ratio N_a/N_c so as to cancel N_c, then $V_a/\pi D_a N_a$ so as to cancel N_a, etc. By an appropriate choice of unity ratios we find that we can change the right member of the equation into an expression involving only the stated permissible quantities. Note the ease, the simplicity and the streamlined appearance of the following:

$$W = \frac{EFV_h}{V_w}\left(\frac{2\pi R_c N_c}{V_h}\right)\left(\frac{N_a}{N_c}\right)\left(\frac{V_a}{\pi D_a N_a}\right)\left(\frac{V_b}{V_a}\right)\left(\frac{\pi D_b N_b}{V_b}\right)\left(\frac{N_d}{N_b}\right)\left(\frac{V_d}{\pi D_d N_d}\right)\left(\frac{V_w}{V_d}\right)$$

$$W = \frac{2R_c D_b FE}{D_a D_d}$$

Another application of unity ratios is the formation of conversion factors for converting a quantity expressed in one set of units to an equivalent quantity expressed in a different set of units.

Suppose we are to convert the quantity, 6 ft., into the corresponding number of inches. We may employ the equation of equivalent quantities 1 ft. = 12 in. From the equation we may form two unity ratios : 1 ft./12 in. = 1 and 1 = 12 in./1 ft. Since these ratios have unity value they may be applied as factors to any quantity without altering the value of that quantity. Note the following : 6 ft. = 6 ft.(12 in./1 ft.) = 72 in. Suppose we invert the conversion factor and obtain 6 ft. = 6 ft.(1 ft./12 in.) = $\frac{1}{2}$ (ft.^{2}/in.). The product quantity is still equivalent to 6 ft. and equivalent to 72 in. Note the following :

$$\frac{1}{2}\frac{ft.^2}{in.} = \frac{1}{2}\frac{ft.^2}{in.}\left(\frac{12\ in.}{1\ ft.}\right)\left(\frac{12\ in.}{1\ ft.}\right) = 72\ in.$$

This technique of making conversions is especially helpful in converting quantities that have compound units. Suppose we wish to convert the power quantity 3 ton miles/hour into the equivalent number of ft. lbs./sec., the equivalent number of horsepower, watts and BTU/min.

$$\frac{3\ ton\ miles}{hr.}\left(\frac{2000\ lb.}{1\ ton}\right)\left(\frac{5280\ ft.}{1\ mile}\right)\left(\frac{1\ hr.}{3600\ sec.}\right) = 8800\ \frac{ft.\ lb.}{sec.}$$

$$8800\ \frac{ft.\ lb.}{sec.}\left(\frac{1\ hp.\ sec.}{550\ ft.\ lb.}\right) = 16\ hp.$$

$$8800\ \frac{ft.\ lb.}{sec.}\left(\frac{30.48\ cm.}{1\ ft.}\right)\left(\frac{1000\ gm.}{2.2\ lb.}\right)\left(\frac{980\ dynes}{1\ gm.}\right)\left(\frac{1\ erg}{1\ dyne\ cm.}\right)$$

$$\left(\frac{1\ joule}{10^7\ ergs}\right)\left(\frac{1\ watt\ sec.}{1\ joule}\right) = 11,948\ watts$$

$$8800\ \frac{ft.\ lb.}{sec.}\left(\frac{1\ BTU}{778\ ft.\ lb.}\right)\left(\frac{60\ sec.}{1\ min.}\right) = 678\ \frac{BTU}{min.}$$

Consider the following problem :

Given : a rectangular trough, having the following dimensions :

Length = 1 yd. = L
Width = 2 ft. = W
Depth = 3 in. = D

Required : the volume of the trough = ? gallons = V

$V = LWD = 1\ yd.(2\ ft.)(3\ in.) = 6\ yd.\ ft.\ in.$

$6\ yd.\ ft.\ in.(3 ft./1\ yd.)(1\ ft./12\ in.) = (3/2)\ ft.^3$

$(3/2)\ ft.^3(7.48\ gal./1\ ft.^3) = 11.22\ gal.$

I have been presenting these applications of unity ratios to my classes for the past twelve years. The techniques that I have illustrated are easy to use and they may be employed in many types of problems.

Long Beach City College, Business and Technology Division

THE PRACTICAL MAN AND THE PURE MATHEMATICIAN:
A MORAL ESSAY

Lyle E. Pursell

Robert E. Machol, a professor of electrical engineering at Purdue, recently told me the following story (of unknown origin):

"A psychologist conducted the following experiment: A man and a woman were stationed on opposite sides of a room. For the first stage of the experiment they were to advance one-half of the way to the center of the room. Next they were to advance one-half of the remaining distance and so on. Now the difference between a pure mathematician and an engineer [a practical man] is that the mathematician would say that they will never meet but the engineer would know that after 5 or 6 stages of the experiment they would be close enough for all practical purposes."

The first reaction of a mathematician to this story is that its author either does not understand the mathematical point of view or has taken a few liberties with the truth for the sake of a story. His "mathematician" is really a man of straw who makes the error of Zeno's Paradox of the Dichotomy assuming that "after infinitely many steps" means "never" and also treats a physical body of considerable size as a point. But more careful analysis of this story shows that the author's "engineer" also makes a common error of the sophomoric mathematician — to take the number of steps as a measure of the accuracy of an approximation by a finite number of steps of an infinite process.

To carry out this analysis we will have to make certain assumptions and consider various possible cases since the story as given, like many statements by non-mathematicians and, alas, some by $(non)^2$-mathematicians, is vague about important details and contains a number of tacit assumptions. First of all what is the couple to do to complete the experiment? Are they to embrace, to shake hands, or to hurl insults at each other? Since the psychologist has deliberately selected two subjects of opposite sex, the last two possibilities seem unlikely. Hence we assume:

(i) The experiment is completed when the couple embrace.

To simplify calculations we also assume :

(ii) Two walls of the room are parallel.

(iii) The subjects start from these walls and move along a common line perpendicular to these two walls.

(iv) They always face the center of the room.

(v) They always stand erect.

As we do not obtain satisfactory results when we treat the bodies of the couple as points, we must select a more accurate method for measuring the distance from one of the subjects to the center of the room. Since the couple start with their backs to the wall it seems reasonable to take the distance of a subject to the center of the room as the distance from the center of the room to the vertical posterior supporting plane of his torso, i. e. a plane parallel to the wall and tangent to his backside.

In this case we see immediately that if we measure the distance using this method and the room is 16 feet across, a common size, then after 5 steps of the experiment the couple will be squeezed between two parallel planes only 6 inches apart which seems entirely too close even for starving undergraduates which are, I am told, the usual human subjects for experiments in psychology. On the other hand if the experiment is carried out in a very large room such as the main room of an armory or fieldhouse 320 feet across, then after 6 steps of the experiment the vertical supporting planes would still be 5 feet apart which is not close enough but after 8 steps the planes would be 15 inches apart which is close enough in my opinion. Additional calculations show that if the distances are measured in this way then the experiment will always end in a finite number of steps but the number of steps required increases without bound with the size of the room.

Of course the psychologist might measure the distance in some other way. For example, he might measure the distance from the center of the room to the anterior supporting plane. Under this assumption after 6 steps of the experiment in a moderately large room 32 feet across the subjects would be 6 inches apart, i. e. there would be two planes 6 inches apart passing between them. Some student deans might like to keep students separated by this amount, but most of the college students whom I have had an opportunity to observe try to get closer together.

Consequently, we see that whether 5 or 6 steps of the experiment are sufficient or not depends upon certain details of the experiment which are not mentioned in the story.

MORAL. In teaching infinite series and other infinite processes in engineering mathematics courses we should place more emphasis on estimating the error after a given number of steps.

Grinnell College and Purdue University

INVARIANCE OF CIRCLE PRODUCT

Ali R. Amir-Moéz

Let the inner product of circles [1], be called more briefly circle product. It is interesting to study the group of transformations which leaves this inner product invariant.

I. Definition: Let C_1 and C_2 be two circles with the following equations:

$$
(1) \qquad a_1(x^2 + y^2) - 2b_1 x - 2c_1 y + d_1 = 0 ,
$$

$$
(2) \qquad a_2(x^2 + y^2) - 2b_2 x - 2c_2 y + d_2 = 0 ,
$$

where all the coefficients and the constant terms are real, x and y are real variables, and at least one of the coefficients is different from zero. We define (C_1, C_2), the inner product of (1) and (2), to be

$$
|a_1|R_1 |a_2|R_2 \cos \alpha
$$

where R_1 and R_2 are respectively the radii of C_1 and C_2 and α is the angle between C_1 and C_2; $|a_1|R_1$ and $|a_2|R_2$ are defined to be the norms of C_1 and C_2 respectively. In order to have real norm we choose real circles.

II. Theorem: Let C_1 and C_2 be given by (1) and (2). Then

$$
(C_1, C_2) = b_1 b_2 + c_1 c_2 - \frac{a_2 d_1 + a_1 d_2}{2} ,
$$

and for

$$
(3) \qquad a(x^2 + y^2) - 2bx - 2cy + d = 0
$$

we have

$$
|a|R = \sqrt{b^2 + c^2 - ad} .
$$

Proof is given in [1].

There are a few interesting and peculier things to observe. In case a real circle is chosen the norm is positive and for $R = 0$ we have zero norm. It is easy to show that the zero norm implies $R = 0$. Thus the norm is zero if and only if $R = 0$, i. e., the points of the plane have zero norm.

Even though later we use an imaginary circle to discuss some properties of circle product we shall not go into discussion of imaginary

circles.

The norm of a straight line $px + qy + d = 0$ is

$$\frac{\sqrt{p^2 + q^2}}{2} .$$

Examples of application of II to analytic geometry were given in [1].

Professor C. C. MacDuffee, who already had shown that (C_1, C_2) was invariant under translation in the plane $x0y$, asked what general transformation in the plane left (C_1, C_2) invariant. Here we answer the question.

III. Note: If we multiply (1) and (2) through by h and k respectively, the inner product will be multiplied by hk. That is, actually we are considering elements of the form

$$a(x^2 + y^2) - 2bx - 2cy + d .$$

IV. A pseudo base: It is easily verified that (1) and (2) can be written as follows:

$$(4) \quad \frac{a_1 + d_1}{2i}[i(x^2 + y^2 + 1)] + \frac{a_1 - d_1}{2}(x^2 + y^2 - 1) + b_1(-2x) + c_1(-2y) = 0 ,$$

$$(5) \quad \frac{a_2 + d_2}{2i}[i(x^2 + y^2 + 1)] + \frac{a_2 - d_2}{2}(x^2 + y^2 - 1) + b_2(-2x) + c_2(-2y) = 0 .$$

Therefore

$$(C_1, C_2) = A_1 A_2 + D_1 D_2 + b_1 b_2 + c_1 c_2 ,$$

where

$$A_1 = \frac{a_1 + d_1}{2i}, \quad A_2 = \frac{a_2 + d_2}{2i}, \quad D_1 = \frac{a_1 - d_1}{2}, \quad \text{and } D_2 = \frac{a_2 - d_2}{2} .$$

Now (C_1, C_2) has the form of the inner product in a four-dimensional real space even though A_1 and A_2 are imaginary.

Let

$$u_1 = i(x^2 + y^2 + 1), \quad u_2 = x^2 + y^2 - 1, \quad u_3 = -2x, \quad u_4 = -2y .$$

Clearly $\{u_1, u_2, u_3, u_4\}$ is an orthonormal base in the sense of circle product. Note that u_1 is not a member of the set of all real circles.

V. Transformations leaving (C_1, C_2) invariant: Any orthogonal transformation of the space spanned by $\{u_1, u_2, u_3, u_4\}$ leaves (C_1, C_2) invariant.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

is orthogonal if

$$\sum_{j=1}^{4} a_{ij}a_{kj} = \delta_{ik}, \quad [2].$$

Note that a_{ij} for all i and j is real in this definition. But we are using this definition for A_1, A_2 regardless of their being imaginary.

We easily see that

$$Au_i = a_{i1}[i(x^2+y^2+1)] + a_{i2}(x^2+y^2-1) - 2a_{i3}x - 2a_{i4}y = v_i, \quad i = 1, 2, 3, 4.$$

The set $\{v_1, v_2, v_3, v_4\}$ is also orthonormal in the sense of circle product.

We know that translation, rotation, homothety, inversion, and symmetry in the plane (conformal group) preserve angle, and transform circle to circle or straight line. So the transformations of the plane which induce orthogonal transformations on the four-dimensional space of the circles will be a combination of these transformations.

VI. Study of transformations: Before discussing the various transformations, we note that the angle is left invariant under transformations mentioned in V. We only have to check the norm of (3) after each transformation. We shall leave it to the reader to verify the following for himself.

(a) Translation:

$$(6) \qquad \begin{cases} x = X + h \\ y = Y + k \end{cases}$$

leaves $|a|R$ invariant. To this transformation corresponds the matrix

$$\begin{pmatrix} \dfrac{h^2+k^2+2}{2} & \dfrac{-i(h^2+k^2)}{2} & -ih & -ik \\[2mm] \dfrac{h^2+k^2}{2i} & \dfrac{2-(h^2+k^2)}{2} & -h & -k \\[2mm] ih & h & 1 & 0 \\[2mm] ik & k & 0 & 1 \end{pmatrix}$$

which is orthogonal according to our definition in V.

(b) Rotation:

$$(7) \qquad \begin{cases} x = (X+h)\cos\alpha - (Y+k)\sin\alpha \\ y = (X+h)\sin\alpha + (Y+k)\cos\alpha \end{cases}$$

leaves $|a|R$ invariant.

(c) Homothety:

$$(8) \qquad \begin{cases} x = hX \\ y = hY \end{cases}$$

changes the norm of (5) to $h^2|a|R$. If we multiply both sides of (3), after using (8) by $1/h^2$, then the norm is preserved.

(d) Inversion:

(9)
$$\begin{cases} x = \dfrac{X}{X^2+Y^2} \\ y = \dfrac{Y}{X^2+Y^2}, & X^2+Y^2 \neq 0 \end{cases}$$

leaves $|a|R$ invariant; so does

(10)
$$\begin{cases} x = \dfrac{X}{X^2+Y^2} - \dfrac{1}{2} \\ y = \dfrac{Y}{X^2+Y^2} - \dfrac{1}{2}, & X^2+Y^2 \neq 0 . \end{cases}$$

The reader may verify that the matrices of the transformations on the four-dimensional space of circle induced by the transformations (7), (8), (9), and (10), are orthogonal.

The whole idea can be generalized for

$$a(x_1^2 + \cdots + x_n^2) - 2a_1 x_1 - \cdots - 2a_n x_n + d = 0 .$$

As a problem it is interesting to write the equation and matrix of conformal group of the plane.

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REMARKS ON THE INVERSE TRIGONOMETRIC AND
HYPERBOLIC FUNCTIONS OF A COMPLEX VARIABLE

L. Pennisi and L. Sjoblom

In general, the standard texts on Complex Variable Theory are not too detailed on the development of the inverse trigonometric and hyperbolic functions. It is the purpose of this classroom note to clarify this development.

By definition, $w = \arcsin z$ if and only if $z = \sin w$, where z and w are complex variables. Now

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i} = \frac{e^{2wi} - 1}{2ie^{iw}},$$

so

$$e^{2wi} - 2ize^{wi} - 1 = 0.$$

Letting $W = e^{wi}$, the above equation becomes

(1) $$W^2 - 2izW - 1 = 0,$$

hence

$$W = iz \pm \sqrt{1 - z^2} = e^{wi}.$$

Taking the logarithm of the above expression, we obtain

$$w = \arcsin z = -i \log [iz \pm \sqrt{1 - z^2}].$$

This is as far as some texts carry the development of $\arcsin z$. However, the above expression for $\arcsin z$ is not entirely satisfactory because:

(a) $\sqrt{1 - z^2}$ is not a well defined symbol and,

(b) even if $\sqrt{1 - z^2}$ were well defined, it is neither continuous nor analytic for all finite values of z.

An expression for $\arcsin z$ will be developed to overcome these objections.

The equation

$$\nu^2 = 1 - z^2$$

has two single valued solutions ν_1 and ν_2 which may be given by

$$\nu_1 = re^{i\theta}, \quad 0 \le \theta < \pi,$$

$$\nu_2 = -\nu_1.$$

In the ensuing discussion, the symbol $\sqrt{1 - z^2}$, for all finite values of z, will be defined as follows:

$$\sqrt{1 - z^2} = \nu_1.$$

Let

(2) $$W_1 = iz + \sqrt{1 - z^2},$$

(3) $$W_2 = iz - \sqrt{1 - z^2},$$

be the roots of equation (1).

Let $\text{Log } z = \text{Log}(re^{i\theta}) = \log r + i\theta, \ -\pi < \theta < \pi$. The principal value of $\log z$ will be denoted by $\text{Log } z$. Observe that $\text{Log } z$ is analytic for all finite values of z except for the set of points along the cut on the negative real axis: $-\infty < x \leq 0$.

From equation (2), we have

(4) $\log W_1 = \log(iz + \sqrt{1 - z^2}) = \text{Log}(iz + \sqrt{1 - z^2}) + 2k_1 \pi i, \quad k_1 = 0, \pm 1, \pm 2, \cdots.$

Since the product of the roots of equation (1) equals the constant term, we obtain

$$W_1 W_2 = -1 = e^{(2k_2 + 1)\pi i}, \quad k_2 = 0, \pm 1, \pm 2, \cdots,$$

hence

(5) $$\log W_2 = -\log W_1 + (2k_2 + 1)\pi i.$$

By equations (4) and (5)

(6) $\log W_2 = \log(iz - \sqrt{1 - z^2})$

$$= -\text{Log}(iz + \sqrt{1 - z^2}) + (2k_3 + 1)\pi i, \quad k_3 = 0, \pm 1, \pm 2, \cdots.$$

Since $W_1 = e^{iw}$, and $w = \arcsin z$, it follows from equation (4) that

$$w = \arcsin z = 2k_1 \pi - i \text{Log}(iz + \sqrt{1 - z^2}).$$

Similarly, from equation (6), we obtain

$$w = \arcsin z = (2k_3 + 1)\pi + i \text{Log}(iz + \sqrt{1 - z^2}).$$

The above two expressions for $\arcsin z$ can be combined into a single equivalent expression,

(7) $\arcsin z = k\pi - (-1)^k i \text{Log}(iz + \sqrt{1 - z^2}), \quad k = 0, \pm 1, \pm 2, \cdots.$

Since $W_1 W_2 = -1$, it follows that neigher W_1 nor W_2 is zero and, in particular, for all finite values of z,

$$W_1 = iz + \sqrt{1 - z^2} \neq 0.$$

Therefore, $iz + \sqrt{1 - z^2}$ is a non-zero single valued function, and so, for a fixed k, equation (7) gives the $\arcsin z$ as a single valued function for all finite values of z. Choosing k even or odd corresponds to choosing respectively W_1 or W_2 as solutions to equation (1).

Without going into detail, it can be shown that the branch $\nu_1 = \sqrt{1 - z^2}$ of $\nu^2 = 1 - z^2$, is a single valued analytic function in an open region R_1, where R_1 consists of the finite z plane except the branch points $z = \pm 1$, and the branch cuts along the real axis given as follows: $-\infty < x \leq -1$, and $1 \leq x < \infty$. Moreover, for all finite z, the function $W_1 = iz + \sqrt{1 - z^2}$

does not take on any negative real values. Therefore, for a fixed k, arc sin z as given by equation (7) is an analytic function in the region R_1.

A similar development for other inverse trigonometric and hyperbolic functions may be given. Let us first introduce the following notation.

The region R_1 denotes the finite z plane except for the set of points on the real axis such that $|x| \geq 1$.

The region R_2 denotes the finite z plane except for the set of points on the imaginary axis such that $|y| \geq 1$.

$$\sqrt{1-z^2} = \nu_1 = r_1 e^{i\theta_1} \qquad 0 \leq \theta_1 < \pi.$$

ν_1^* is defined on R_1 and such that ν_1^* equals ν_1 on R_1.

$$\sqrt{z^2-1} = \nu_2 = r_2 e^{i\theta_2} \qquad 0 \leq \theta_2 < \pi.$$

ν_2^* is defined on R_1 and such that ν_2^* equals ν_2 on R_1.

$$\sqrt{z^2+1} = \nu_3 = r_3 e^{i\theta_3} \qquad 0 \leq \theta_3 < \pi.$$

ν_3^* is defined on R_2 and such that ν_3^* equals ν_3 on R_2.

We shall summarize the above results for arc sin z in Theorem 1 below and state corresponding theorems for other inverse trigonometric and hyperbolic functions.

THEOREM 1. *For all finite z,*

$$\text{arc sin } z = k\pi - (-1)^k i \operatorname{Log}(iz+\sqrt{1-z^2}), \quad k = 0, \pm 1, \pm 2, \cdots.$$

Moreover, for z in R_1 and k fixed, arc sin z is analytic and has the derivative

$$\frac{d}{dz}(\text{arc sin } z) = \frac{(-1)^k}{\sqrt{1-z^2}},$$

where k is chosen as follows: k is even or odd when we take the branch ν_1^* or $-\nu_1^*$ respectively.

THEOREM 2. *For all finite z,*

$$\text{arc cos } z = 2k\pi \mp i \operatorname{Log}(z+\sqrt{z^2-1}), \quad k = 0, \pm 1, \pm 2, \cdots.$$

Moreover, for z in R_1 and k fixed, arc cos z is analytic and has the derivative

$$\frac{d}{dz}(\text{arc cos } z) = \mp \frac{i}{\sqrt{z^2-1}},$$

where the $\mp$ sign is chosen as follows: the branch ν_2^* is associated with the $-$ sign of the arc cos z and its derivative, and the branch $-\nu_2^*$ is associated with the $+$ sign of the arc cos z and its derivative.

THEOREM 3. *For all finite z except $z = \pm i$,*

$$\text{arc tan } z = k\pi + \frac{1}{2i} \operatorname{Log}\left[\frac{1+iz}{1-iz}\right], \quad k = 0, \pm 1, \pm 2, \cdots.$$

Moreover, for z in R_2 and k fixed, arc tan z is analytic and has the derivative

$$\frac{d}{dz}(\text{arc tan } z) = \frac{1}{1+z^2}.$$

THEOREM 4. *For all finite z,*

$$\text{arc sinh } z = k\pi i + (-1)^k \text{Log}(z+\sqrt{z^2+1}), \quad k = 0, \pm 1, \pm 2, \cdots.$$

Moreover, for z in R_2 and k fixed, arc sinh z is analytic and has the derivative

$$\frac{d}{dz}(\text{arc sinh } z) = \frac{(-1)^k}{\sqrt{z^2+1}},$$

where k is chosen as follows : k is even or odd when we take the branch ν_3^* or $-\nu_3^*$ respectively.

THEOREM 5. *For all finite z,*

$$\text{arc cosh } z = 2k\pi i \pm \text{Log}(z+\sqrt{z^2-1}), \quad k = 0, \pm 1, \pm 2, \cdots.$$

Moreover, for z in R_1 and k fixed, arc cosh z is analytic and has the derivative

$$\frac{d}{dz}(\text{arc cosh } z) = \pm \frac{1}{\sqrt{z^2-1}},$$

where the $\pm$ sign is chosen as follows : the branch ν_2^* is associated with the + sign of the arc cosh z and its derivative, and the branch $-\nu_2^*$ is associated with the $-$ sign of arc cosh z and its derivative.

THEOREM 6. *For all finite z except $z = \pm 1$,*

$$\text{arc tanh } z = k\pi i + \frac{1}{2}\text{Log}\left(\frac{1+z}{1-z}\right), \quad k = 0, \pm 1, \pm 2, \cdots.$$

Moreover, for z in R_1 and k fixed, arc tanh z is analytic and has the derivative

$$\frac{d}{dz}(\text{arc tanh } z) = \frac{1}{1-z^2}.$$

University of Illinois
Navy Pier, Chicago, Ill.

A NOTE ON THE WELL-ORDERING OF SETS

E. S. Keeping

Having experienced some difficulty with the treatment of ordering in M. E. Munroe's "Measure and Integration", usually a very clearly-written text, the author suggests that a note on this subject might possibly be helpful to others.

A set of objects is *ordered* if there is a relation of precedence established between the members of the set, so that of any two distinct members we can always say which one precedes the other. The relation "x precedes y" will be written $x \prec y$, or equivalently $y \succ x$, read as "y follows x". It seems to me unfortunate to use the notation $x < y$ (as Munroe does) because of the possible implication that the only type of ordering considered is that in increasing order of size.

The precedence relation must be transitive, so that if $x \prec y$ and $y \prec z$, then $x \prec z$. The set of all integers is ordered, whether arranged in the natural order $\cdots -3, -2, -1, 0, 1, 2, 3 \cdots$ or in the order $0, 1, -1, 2, -2, 3, -3, \ldots$. In the first case the relation $\prec$ is the same as $<$, in the second case not.

Any finite set of n objects can be arbitrarily ordered in $n!$ ways, and the empty set can be conventionally regarded as ordered. In any ordering of a finite set there is a first object which precedes all the others and a last object which follows all the others. This is not necessarily true of infinite sets.

The set of integers in the natural order has neither a first nor a last element, although when arranged in the second way given above it has a first element. The set of all rational numbers between 0 and 1 (not inclusive) has no first or last element when arranged in order of size since, for example, no matter how small a rational positive number we select, we can always halve it to get a still smaller one. But this set can be put in an order which does have a first element, thus :

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \cdots$$

where the fractions are arranged in order of their denominators, and, for each denominator, in the order of their numerators, all fractions not in their lowest terms being omitted. This ordering is clearly not according to size.

An ordered set is said to be *well-ordered* if every non-empty sub-set

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has a first element. The elements in the sub-set must of course preserve the ordering in the original set (this is not explicitly stated by Munroe). Also Munroe states that the sub-set must have a "least" element, but "least" in this sense does not bear its ordinary meaning. The set of integers in the natural order is not well-ordered, because the sub-set (for example) of all integers less than 0 has no first element. But in the re-ordering 0, 1, -1, 2, -2, 3, -3 $\cdots$, this sub-set now has a first element, namely -1, which is actually the greatest of the lot. The set therefore is now well-ordered.

The set $\{n - \frac{1}{k}\}$, where $n = 1, 2, 3 \cdots$ and, for each n, $k = 1, 2, 3 \cdots$, may be written out as follows:

$$0, \ \frac{1}{2}, \ \frac{2}{3}, \ \frac{3}{4} \cdots 1, \ 1\frac{1}{2}, \ 1\frac{2}{3}, \ 1\frac{3}{4} \cdots 2, \ 2\frac{1}{2}, \ 2\frac{2}{3} \cdots$$

and this is a well-ordering according to increasing size. The set $\{n + \frac{1}{k}\}$, however, becomes when written out:

$$2, \ 1\frac{1}{2}, \ 1\frac{1}{3}, \ 1\frac{1}{4} \cdots 3, \ 2\frac{1}{2}, \ 2\frac{1}{3}, \ 2\frac{1}{4} \cdots 4, \ 3\frac{1}{2}, \ 3\frac{1}{3} \cdots$$

in which the well-ordering is not according to size. If the set is re-arranged in order of increasing size it ceases to be well-ordered. The sub-set of fractions of the form $1 + \frac{1}{k}$, for example has no least element.

Every ordering of a finite set clearly gives a well-ordered set. We have seen that some ordered infinite sets are not well-ordered. The question naturally arises whether every ordered set can be re-arranged (if necessary) so as to be well-ordered, and the answer is yes. At first sight the question may seem trivial, since we can always pick out some element of the given set M and call it the first one (say m_1), then any element of the remaining set $M - m_1$ (say m_2), and so on. This process can apparently be continued until M is exhausted and the set is then well-ordered.

However, as Zermelo first showed, the matter is not really as simple as this. The set M may contain many limit points, perhaps more than a denumerable number of them, and an axiom of choice is required to permit the continued application of the process of picking out elements. A good account of Zermelo's proof of the well-ordering theorem may be found in Kamke's little book on "The Theory of Sets".

The basic assumption in this proof is that if N is any non-empty sub-set of M, we can always distinguish one particular element n of N by means of a functional relationship $n = f(N)$. This is where the axiom of choice comes in. It is not of course necessary (or in general possible)

that distinct sub-sets should have distinct elements n.

The proof runs in a series of steps. It is first shown that *well-ordered* non-empty sub-sets of M can be picked out according to a particular rule. If c is any element of such a sub-set Γ, and if Γ_c is the set of elements of Γ which *precede* c, then c itself obviously belongs to the set $M - \Gamma_c$, and is chosen as the distinguished element of that set. In the notation above, $c = f(M - \Gamma_c)$. If m_1 is the distinguished element for the whole set M, it follows that m_1 must be the *first* element of any such Γ-set. (The set Γ_{m_1} is empty, and $m_1 = f(M) = f(M - \Gamma_{m_1})$.) A similar argument shows that of any two distinct Γ-sets one is always a segment of the other, so that the longer one starts off with all the elements of the shorter one in the same order.

It is then proved that the union Σ of all Γ-sets is well-ordered and itself constitutes a Γ-set. Finally it is shown that Σ must coincide with M itself, which is therefore well-ordered.

It must be clearly understood that the ordering relation by no means implies that the members of the set can be put into a one-one relation with the integers 1, 2, 3 ···. This happens to be true for some of the examples given above, but the relation is much more general. The numbers in the closed segment of the real axis, $0 \leq x \leq 1$, are well-ordered according to size, but they are not denumerable.

Although, according to the well-ordering theorem, the set consisting of the whole continuum *can* be well-ordered, the theorem provides no explicit way of accomplishing this.

University of Alberta
Edmonton, Alberta, Canada

CURRENT PAPERS AND BOOKS

Edited by H. V. Craig

NOTE ON A PAPER OF STEINBERG

H. W. Gould

Introduction. The purpose of this paper is to examine the two algebraic identities.

$$\sum_{k=1}^{m} \frac{m!\, k m^{n-k-1}}{(m-k)!} = m^n, \tag{1}$$

$$\sum_{k=1}^{n-1} \frac{m!\, k m^{n-k-1}}{(m-k)!} + \frac{m!}{(m-n)!} = m^n, \tag{2}$$

for $m > n$.

derived recently by Donald Steinberg, [1], and prove a more general identity which includes these and others as easy special cases. Essentially we shall remove the restriction that m be an integer. This can be effected by writing the relations in a slightly different form, using the notation of binomial coefficients.

Thus we may write relation (2) above in the form

$$\sum_{k=0}^{n-1} \binom{m}{k} k \cdot k! \cdot m^{n-k-1} = m^n - \frac{m!}{(m-n)!},$$

or finally

$$\sum_{k=0}^{n} \binom{m}{k} \frac{k \cdot k!}{m^{k+1}} = 1 - \binom{m}{n+1} \frac{(n+1)!}{m^{n+1}}, \tag{3}$$

In this we see that there is a strong resemblance between the summand on the left and the expression on the right. In fact there is a strong resemblance between relation (3) and the familiar identity

$$\sum_{k=0}^{n} k \cdot k! = (n+1)! - 1,\tag{4}$$

In fact if relation (3) were true for all real values of m we could deduce (4) from it. We recall that the binomial coefficient $\binom{x}{k}$ is a polynomial of degree k in x, and satisfies such relations as

$$\binom{x}{k} = (-1)^k \binom{-x+k-1}{k},\tag{5}$$

$$\binom{x+1}{k} = \binom{x}{k} + \binom{x}{k-1},\tag{6}$$

$$\binom{x}{k} = (-1)^k \prod_{j=1}^{k} \left(1 - \frac{x+1}{j}\right),\tag{7}$$

In particular it follows from (7), with $x = -1$, that

$$\binom{-1}{k} = (-1)^k,\tag{8}$$

If we make use of this fact and allow $m = -1$ in relation (3) we would immediately obtain relation (4). Thus there is some motivation here for expecting a generalization.

Looking back over some notes on matters such as these, the writer finds that one of his own derivations of a generalization of relation (4) depends upon the following principle:

$$\sum_{k=0}^{n} \{f(k+1) - f(k)\} = f(n+1) - f(0) = \sum_{k=0}^{n} \Delta f(k),\tag{9}$$

What we shall do here is write down an expression for a function which when substituted in (9) will yield the desired generalization.

We make the following definition for our function:

$$f(k) = \binom{x}{k}^p \frac{k!^p}{x^{kp}},\tag{10}$$

where x is real and p is any positive or negative integer. From this it follows that

$$\Delta f(k) = f(k+1) - f(k)$$

$$= \frac{x!^p}{(x-k-1)!^p x^{(k+1)p}} - \frac{x!^p}{(x-k)!^p x^{kp}}$$

$$= \frac{x!^p}{(x-k)!^p x^{(k+1)p}} \{(x-k)^p - x^p\}$$

$$= \binom{x}{k}^p \frac{k!^p}{x^{(k+1)p}} \{(x-k)^p - x^p\},\tag{11}$$

Applying our summation principle, (9), we therefore find that

$$\sum_{k=0}^{n} \binom{x}{k}^p \frac{k!^p}{x^{(k+1)p}} \{(x-k)^p - x^p\} = \binom{x}{n+1}^p \frac{(n+1)!^p}{x^{(n+1)p}} - 1 , \tag{12}$$

and this is our generalized identity which we wished to exhibit.

From (12), with $p = 1$ we obtain relation (3) again with x written instead of m. Also, relation (1) follows from (3) when we set $n = m$ since we then observe that $\binom{m}{m+1} = 0$, for m an integer ≥ 0.

Finally we observe that when $p = -1$ we obtain the relation

$$\sum_{k=0}^{n} \frac{x^{k+1} \cdot k}{x(x-k)k!\binom{x}{k}} = \frac{x^{n+1}}{(n+1)!\binom{x}{n+1}} - 1 , \tag{13}$$

which is a generalization of the familiar identity

$$\sum_{k=0}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!} , \tag{14}$$

which follows from (13) when we set $x = -1$ and again apply relation (8).

If we let $x = m = \text{integer} \geq n+1$, then (13) may be written in the form

$$\sum_{k=0}^{n} km^k (m-k-1)! = m^{n+1}(m-n-1)! - m! , \tag{15}$$

This relation is a companion-piece to Steinberg's relation. We raise the following question then: Through what combinatorial considerations of the type Steinberg applied, would this relation arise as a natural consequence?

One final remark is that we could set $x = -\frac{1}{2}$ and such like values in relation (12) and apply formulas for $(-\frac{1}{2})!$ in order to obtain other interesting special cases.

There is no reason why we could not alter the definition (10) somewhat and go in some entirely different direction. But it takes some insight to see ahead what form $\Delta f(k)$ will assume.

It is felt that the above considerations throw a different light upon the derivation of identities and exhibit a different aspect of their structure and content. We do not believe (12) is unknown to the vast literature on this subject.

BIBLIOGRAPHY

1. Donald A. Steinberg, *Combinatorial Derivations of Two Identities*, Mathematics Magazine, Vol. 31, No. 4, pages 207-9, March-April, 1958.

West Virginia University
Morgantown, West Virginia

Mathematics in Fun and in Earnest. By Nathan A. Court. Dial Press, Inc., Nr w York, 1958, 250 pages. $4.75.

It is always a pleasure to read a book or paper from the gifted pen of Nathan Altshiller Court. His credo, mathematics in earnest should be fun, mathematics in fun may be earnest, is delightfully exemplified in this book, which belongs on the shelf, not only of every college and every college mathematician, but also every high school library. In spite of the light touch which Dr. Court maintains, a great deal of serious mathematics is made available to the non-professional in this little volume. It would be picayune to mention the misprints, which no doubt the publisher will have removed in the second printing anyway. This is a highly enjoyable book, and it is a pleasure to recommend it to you.

Richard V. Andree
The University of Oklahoma

Guide to the Literature of Mathematics and Physics. By Nathan Grier Parke III. Dover Publications, 1958, 436 pages. $2.49.

The *Guide* includes an up-to-date listing of agencies and individuals who are engaged in Russian translation programs. It is one of the many features of this unusual work which was written as an aid for people embarking on research in physics, mathematics, and related engineering sciences.

The *Guide* lists more than 5,500 key works under 120 subject headings such as Projective Geometry, Geometric Optics, and Cosmic Rays. The titles are carefully cross-referenced and the subject classifications are based on those used by *Mathematical Reviews*, making it easy to build your own bibliography. Whenever a foreign language title appears, any translations are also listed. If there is no English translation of a Russian work, for example, the *Guide* will tell you if there is one in German.

Extensive lists of indexes, documentary reproductions, abstracts and other bibliographic aids have been included to give the *Guide* maximum effectiveness as a research tool for students and professionals.

Gil Tauber

EQUIVALENCE SYMBOL AND PARENTHESES SYMBOLS

Incorporated, New York, 1959, p. 205. Karl Menger found the conditions under which the present notation is meaningful for our purposes. Cf. his *Basic Concepts of Mathematics*, Ill. Inst. of Tech., Chicago, 1957, pp. 26-29. I am indebted to Professor Menger for calling my attention to all these references.

The University of Texas
Austin, Texas

Editor's Note

In the future, author's proofs will not be sent outside of North America. Corrections on proofs are too rare to justify the usual delay in such cases.

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton

PROPOSALS

383. *Proposed by Raphael T. Coffman, Richland, Washington.*

Cut any square into not more than six pieces which can be reassembled to form a cube having its surface area equal to the area of the square. Bending of the pieces is permissible.

384. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

Let (a_{ij}) be a matrix of nth order the sum of the elements of whose rows equals 1. Prove that the totality $[(a_{ij})]$ form a group of infinite order.

385. *Proposed by David L. Silverman, Greenbelt, Maryland.*
Find, if they exist:
I. The smallest cube which is the sum of three primes, the sum of any two of which is a square.
II. The largest square which is the sum of three primes, the sum of any two of which is a cube.

386. *Proposed by Leo Moser, University of Alberta.*
Let $a_i \geqq 0$ and

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n .$$

Let

$$g(x) = f^2(x) = b_0 + b_1 x + \cdots + b_{2n} x^{2n} .$$

Prove that

$$b_{2r+1} \leq \frac{1}{2} f^2(1) .$$

387. *Proposed by D. S. Mitrinovitch, University of Belgrade, Yugoslavia.*
Prove the relation,

$$\left[\frac{\partial^n}{\partial t^n} \left(\frac{1}{1-t} e^{\frac{-xt}{1-t}} \right) \right]_{t=0} = e^x \frac{d^n}{dt^n} (x^n e^{-x})$$

n a natural number, by induction.

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388. *Proposed by M. S. Krick, Albright College, Pennsylvania.*
Prove that

$$\binom{n}{k} = \sum_{s=0}^{t} \binom{t}{s}\binom{n-t}{k-s}, \quad n-t \geq k \geq t$$

389. *Proposed by B. Keshava R. Pai, Belgaum, India.*

There were five Wednesdays in the month of February, 1956, a leap year. During the subsequent century, which years will have five Wednesdays in February?

SOLUTIONS

Late Solutions

349. *Robert Woolery and Stanley Long (jointly).*
357, 358. *D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts.*

The Wizard

362. [January 1959] *Proposed by David L. Silverman, Greenbelt, Maryland.*

Not having colored ink with which to dot the foreheads of his three apprentices, the wizard wrote numbers on their foreheads instead and told the apprentices that each had been given a prime number, the three of which formed the sides of a triangle with prime perimeter. The apprentice who deduced his number first was to be the wizard's successor. Apprentice A was given a 5 and B a 7. After a few minutes of silence C was able to deduce his number. What was it?

Solution by Clarence M. Sidlo, Framingham, Massachusetts.

C reasons thusly:

My number must be prime, not equal to or greater than the sum of the other two sides of the triangle nor equal to or less than the difference of the other two sides, hence it is less than or equal to 11 and greater than or equal to 3, and further must sum with $5 + 7 = 12$ to make a prime number. Therefore, my number is 5, 7, or 11.

Now if my number is 5, B will see that both A and I have 5's, and will deduce that he must have a 1, 3, or 7. He knows at once that his number is not 1, since either A or I would have deduced immediately that our numbers were identical. If B assumes that he has a 3, then he will know that I must guess that I have a 3 or a 5. He will go on to suppose that if I then assume that I have a three, I would realize in turn that A would know he could only have a 5, or a 1, of course. But again, the same reasoning holds. Since A has not indicated this solution, and I have not taken advantage of this fact to indicate that I have a 5, B would conclude that he does not have a 3, and so must have a 7. Since B has not done this, my number is not 5.

If my number is 7, A will see that B and I both have 7's, and will deduce that he must have a 1, 3, or 5. By the same reasoning as B above, A will know that his number is not 1. If A assumes that he has a 3, then he realizes that the only conclusion I can come to is that I have a 7. Since I have not done this, A would deduce that he does not have a 3, and so must have a 5. Since A has not indicated this, my number is not 7.

Therefore, my number is 11.

Also solved by Joan M. Connell, Needham, Massachusetts; James C. Ferguson, University of Washington; Lynn Knighten, Palos Verdes, California; Joseph D. E. Konhauser, Haller, Raymond and Brown, Inc., State College, Pennsylvania; Joseph M. Synnerdahl, Canisius College; P. D. Thomas, U.S. Coast and Geodetic Survey, Washington, D.C.; C.W. Trigg, Los Angeles City College; Dale Woods, Idaho State College; and the proposer.

A Maximum Angle

363. [January 1959] *Proposed by Brian Brady, Richmond, N.S.W., Australia.*

OAB is a line and P is a point which moves along a line through O which makes an angle α with OAB. If $OA = a$, $AB = b$ find the maximum value of angle APB.

I. *Solution by P. D. Thomas, U.S. Coast and Geodetic Survey, Washington, D.C.*

Let OP, AP, BP be t, u, v respectively and let angle $APB = \beta$. Then by the law of cosines

$$v^2 = t^2 + (a+b)^2 - 2t(a+b)\cos\alpha$$

$$u^2 = t^2 + a^2 - 2at\cos\alpha \tag{1}$$

$$\cos\beta = (u^2 + v^2 - b^2)/2uv, \quad \cot\beta = (\cos\beta)/1 - \cos^2\beta)^{1/2}$$

$$\cot\beta = (u^2 + v^2 - b^2)/[2b^2(u^2 + v^2) - (u^2 - v^2)^2 - b^4]^{1/2} \tag{2}$$

The values from (1) placed in (2) give

$$\cot\beta = 2[t^2 - t(2a+b)\cos\alpha + a(a+b)]/2bt\sin\alpha$$
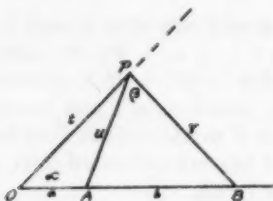$$= [t - (2a+b)\cos\alpha + a(a+b)/t]/b\sin\alpha \tag{3}$$

$$d(\cot\beta)/dt = [1 - a(a+b)/t^2]/b\sin\alpha = 0$$

whence
$$t = (a^2 + ab)^{1/2} . \tag{4}$$

The value of t from (4) placed in (3) gives

$$\beta = \operatorname{arccot} (1/b)[2(a^2+ab)^{1/2}\csc\alpha - (2a+b)\cot\alpha]. \tag{5}$$

Note from (4) that the distance t which minimizes $\cot\beta$ (maximizes angle APB) is independent of the angle α. For $\alpha = 90°$, (5) gives $\beta = \operatorname{arccot} 2(a^2 + ab)^{1/2}/b$.

II. *Solution by the proposer.*

Consider the circle through A, B and tangent to the line OP at R, say. If P' is any other point on OP, and AP' meets the circle in R', then angle $AP'B <$ angle $AR'B =$ angle ARB. Hence R is the position of P for which angle APB is a maximum.

If $OR = x$ we have $x^2 = a(a+b)$

also

$$\text{angle } ORA = \pi/2 - \tfrac{1}{2}(\alpha + \theta)$$

$$\text{angle } OAR = \pi/2 - \tfrac{1}{2}(\alpha - \theta)$$

from OAR

$$\frac{x}{a} = \frac{\cos \tfrac{1}{2}(\alpha - \theta)}{\cos \tfrac{1}{2}(\alpha + \theta)}$$

whence

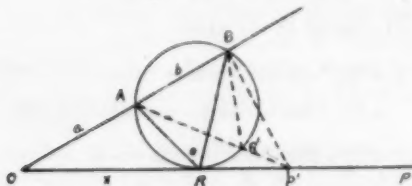$$\frac{x-a}{x+a} = \tan \alpha/2 \tan \theta/2$$

$$\tan \theta/2 = \frac{x-a}{x+a} \tan \alpha/2$$

hence

$$\tan \theta = \frac{2(x^2 - a^2) \cot \alpha/2}{(x+a)^2 - (x-a)^2 \cot^2 \alpha/2}$$

since $x^2 = a(a+b)$ this reduces to the answer

$$\theta = \arctan \left[\frac{\sin \alpha}{2\sqrt{a(a+b)} - (2a+b) \cos \alpha} \right]$$

Also solved by *Raphael T. Coffman, Richland, Washington; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; Joseph D. E. Konhauser, Haller, Raymond and Brown, Inc., State College, Pennsylvania; Catherine A. O'Connor, Regis College, Massachusetts; F. D. Parker, University of Alaska; S. W. Poorbough, University of Virginia; Abdel S. Said, Brooklyn Polytechnic Institute, Robert E. Shafer, University of California Radiation Laboratory; A. Smith, W. Sims and R. Gwilliam (jointly), Ryerson Institute of Technology, Toronto, Ontario, Canada; Walter R. Talbot, Jefferson*

City, Missouri; and Dale Woods, Idaho State College.

Euler's Limit e

364. [January 1959] *Proposed by Barney Bissinger, Lebanon Valley College, Pennsylvania.*

Find

$$\lim_{x \to \infty} (x \tan 1/x)^{3x^2}$$

Solution by Robert E. Shafer, University of California Radiation Laboratory.

A non rigorous method of solution would be to expand $x \tan 1/x$ as a power series so that

$$\lim_{x \to \infty} (1 + 1/3x^2 + 2/15x^4 + \cdots)^{3x^2} \to e$$

from Euler's limit. To guarantee this limit in a rigorous fashion, we would have for some $x > N$

$$(1 + 1/3x^2) < (x \tan 1/x) < \frac{1}{\sqrt{1-2/3x^2}}$$

so that by our limit process, we would have

$$\lim_{x \to \infty} (1 + 1/3x^2)^{3x^2} \leq \lim_{x \to \infty} (x \tan 1/x)^{3x^2} \leq \lim_{x \to \infty} (1-2/3x^2)^{-3x^2/2}$$

each extreme limit of which approaches e.

Also solved by D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; Robert Kilmoyer, Lebanon Valley College, Joseph D. E. Konhauser, Haller, Raymond and Brown, Inc., State College, Pennsylvania; M. Morduchow, Polytechnic Institute of Brooklyn; Abdel S. Said, Polytechnic Institute of Brooklyn; W. R. Talbot, Jefferson City, Missouri; P. D. Thomas, U.S. Coast and Geodetic Survey, Washington, D. C.; Dale Woods, Idaho State College; and the proposer.

An Infinite Hexagonal Lattice

365. [January 1959] *Proposed by Robert E. Shafer, University of California Radiation Laboratory.*

Find the cartesian coordinates of all points in an infinite hexagonal lattice, given the circumscribed circle radius of a hexagonal cell as unit length.

Solution by D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts.

Orienting the coordinate axes so that one hexagon has vertices at

$$(\pm 1, 0), \quad (\pm \frac{1}{2}, \pm \frac{\sqrt{3}}{2})$$

it is easily **seen** that any point in the lattice has coordinates (x, y) in

one of the two forms

$$(\pm n, \pm k\sqrt{3}/2) \quad [k \text{ even}, \ n \not\equiv 0 \,(\text{mod } 3)]$$

$$(\pm (n+\tfrac{1}{2}), \pm k\sqrt{3}/2) \quad [k \text{ odd}, \ n \not\equiv 1 \,(\text{mod } 3)]$$

where n and k are non-negative integers.

Also solved by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; Joseph D. E. Konhauser, Haller, Raymond and Brown, Inc., State College, Pennsylvania; F. D. Parker, University of Alaska; and the proposer.

Roots of $e^z = z$

366. [January 1959] *Proposed by George Bergman, Stuyvesant High School, New York.*

Show that an infinite number of complex numbers z satisfy the equation $e^z = z$.

I. *Solution by Walter R. Talbot, Jefferson City, Missouri.*

Let $z = x + iy$ where x and y are real. Then

$$x + iy = e^z = e^x \cdot e^{iy} = e^x(\cos y + i \sin y).$$

Then $x = e^x \cos y$

and $y = e^x \sin y$

so that $x^2 + y^2 = e^{2x}$ or $y = \pm\sqrt{e^{2x} - x^2}$

It is sufficient to show the radicand is non-negative for an infinite number of values of x.

By logarithms we know $e^{2x} > 2e^x$ if $x > \ln 2$, and by the infinite series for e^x, we know $2e^x > x^2$ if $x > 0$. Then $e^{2x} > 2e^x > x^2$ for $x > \ln 2$. Under these conditions y is real for an infinite number of real values of x.

II. *Solution by Dale Woods, Idaho State College.*

$$e^z = z$$

$$e^{x+iy} = x + iy$$

$$e^x(\cos y + i \sin y) = x + iy \,.$$

Therefore $x = e^x \cos y$

$$y = e^x \sin y \quad \text{or} \quad x = \ln(y \csc y)$$

Hence $\ln(y \csc y) = y \cot y$

Now let $f(y) = \ln(y \csc y) - y \cot y$

$$f(\pi/2 + 2n\pi) = \ln(\pi/2 + 2n\pi) > 0, \quad n \text{ a natural number}$$

$$f(\pi/4 + 2n\pi) = [\ln\sqrt{2}(\pi/4 + 2n\pi)] - (\pi/4 + 2n\pi) < 0$$

Since $f(y)$ is continuous in each interval between $\pi/4 + 2n\pi$ and $\pi/2 + 2n\pi$ for each n, there are an infinite number of roots of the equation and hence values of y (and hence x and z) that satisfy the respective equations.

Also solved by D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; James C. Ferguson, University of Washington;

and the proposer.

A Monotone Function

367. [January 1959] *Proposed by J. B. Love, Eastern Baptist College, Pennsylvania.*

Let $\phi(x)$ and $\psi(x)$ be monotone nondecreasing functions as $x \to \infty$ and both be positive for $0 < x < \infty$. Let $f(x)$ be defined for $x > 0$ with $f'(x)$ and $f''(x)$ existing such that $|f(x)| < \phi(x)$ and $|f''(x)| < \psi(x)$. Show that

$$|f'(x)| < 2[\phi(x) \cdot \psi(x)]^{\frac{1}{2}}$$

Solution by Robert Kilmoyer, Lebanon Valley College.

Since for $x > 0$ the quadratic polynomial in x

$$|f''(x)|x^2 + |f'(x)|x + |f(x)|$$

is non-negative, it follows that

$$\psi(x)x^2 + |f'(x)|x + \phi(x) > 0$$

and the condition for the quadratic in the above inequality to have no real roots as dictated by the strict inequality is

$$|f'(x)|^2 - 4\psi(x) \cdot \phi(x) < 0$$

the desired result.

Also solved by the proposer.

Comment on Problem 354

354. [September 1958 and March 1959] *Proposed by Lowell Van Tassel, San Diego Junior College, California.*

Comment by Melvin Bloom, Miami University, Ohio. The published solution proceeds under the assumption that the axis of rotation is perpendicular to the path of the arrow. Assuming random direction of the axis, P_1, the probability of piercing the black exactly once is:

$$P_1 = \frac{2}{\pi} \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin\phi \cos\phi \, d\phi = \frac{1}{2}$$

By symmetry P_2, the probability of piercing black twice, is $\frac{1}{2}(1 - P_1)$ and the required probability is the sum of these two:

$$P = P_1 + P_2 = 3/4$$

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 253. An isosceles triangle with sides x, x and a is equal in area to one with sides x, x and b, where $a \neq b$. Show that only one solution exists for x without using Hero's formula. [*Submitted by David L. Silverman*]

Q 254. Prove that $\displaystyle\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
    [*Submitted by M.S.Klamkin*]

Q 255. Find a number which when reduced by 7 and the remainder is multiplied by 7 gives the same result as when the number is reduced by 11 and the remainder is multiplied by 11. [*Submitted by C. W. Trigg*]

Q 256. Siven a right triangle ABC with legs a and b. Find the lengths of the angle trisectors to the hypotenuse. [*Submitted by Jeff Cheeger*]

Q 257. Solve in integers $x^n + y^n = 2^{n+1}$. [*Submitted by V. F. Ivanoff*]

Answers

A 257. Let $a^n + b^n = c$, a and b being arbitrary integers. Then $a^n c^n + b^n c^n = c^{n+1}$.

Similarly
$$y = \frac{2ab}{b\sqrt{3} + a}.$$

so that
$$x = \frac{2ab}{a\sqrt{3} + b}.$$

A 256. Since the area of triangle ABC equals the sum of the triangles formed by the trisectors; ACD, DCE, and ECB, we have:
$$ab/2 = x/2(a \sin 60^\circ + b \sin 30^\circ)$$

A 255. The common result must have 7 and 11 as factors, thus the number is $7 + 11 = 18$. The method is general, since the solution of $(x-7)7 = (x-m)m$ is $k + m$.

A 254. Let $a + b - x = y$. Then we have $\int_a^b f(a+b-x)dx = -\int_b^a f(x)dx$.

A 253. Reflecting the right triangle with legs $a/2$ and $b/2$ through each of its legs gives the desired pair of isosceles triangles. Hence $x = \frac{1}{2}\sqrt{a^2 + b^2}$.

Comments on the Quickies

Q 232. [November 1958] *Comments by William E. F. Appuhn.*
The question was to find an integral root of

$$360 = (x-2)(x-3)(x-4)(x-5)$$

This equation has two integral roots, one positive and one negative. The solution submitted in the current issue has two disadvantages. First, it will not allow of a negative solution since it involves $x!$ and is therefore restricted to positive integral values of x. Second, while it is interesting, it is longer and more involved than necessary, as will be evident upon consideration of the following solution.

Noting that the product of four consecutive (non-zero) integers lies between the fourth powers of its middle factors and that 360 lies between $(\pm 5)^4$ and $(\pm 4)^4$, it immediately follows that these middle factors are either 5 and 4, or, -4 and -5, giving $x = 8$ or $x = -1$, as two quicker quickie solutions.

Q 242. [March 1959] *Comment by D. M. Long.*
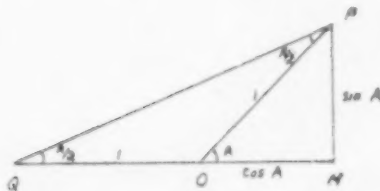The printed solution is not the simplest as it introduces the symbol

[] where strictly algebraic functions would have sufficed. I suggest $f(n) = \dfrac{3 - (-1)^n}{4}$. As a quickie one could ask the general question, find an $f(n)$ such that $f(\text{even}) = a$ and $f(\text{odd}) = b$. The solution is
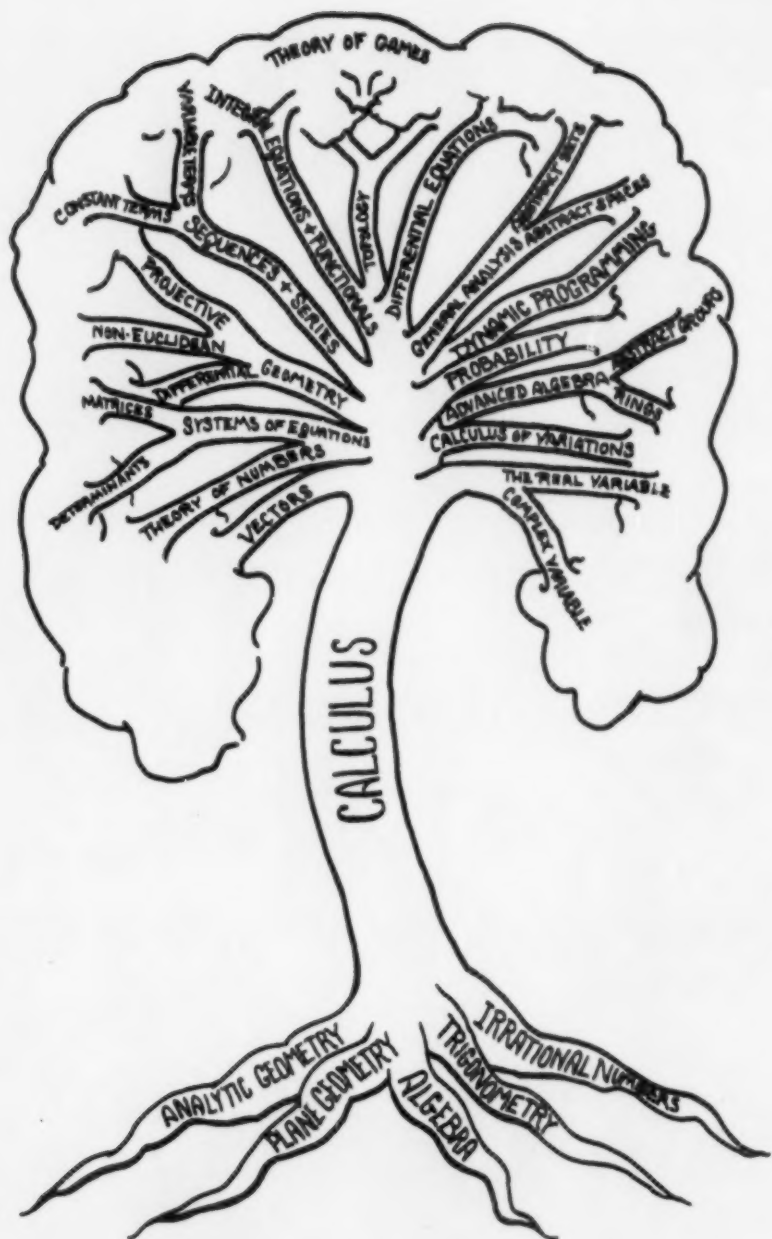
$$f(n) = \frac{[1 + (-1)^n]a + [1 - (-1)^n]b}{2}.$$

Q 243. [March 1959] *Comment by Norman Anning.*
Extend MO to Q making $OQ = OP$. Then we have

$$\tan\frac{A}{2} = \frac{MP}{QM} = \frac{MP}{QO + OM} = \frac{\sin A}{1 + \cos A}.$$

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